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A SYLLABUS

OF

A COURSE OF LECTURES

UPON

TRIGONOMETRY,

AND

THE APPLICATION OF ALGEBRA

TO

GEOMETRY.

2.

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ERRATA.

PAGE	LINE	ERROR.	CORRECTION.
14	6 from bottom,	forces	forms.
20	11 …… top,	or	as.
34	3 …… bottom,	$\sin^3 \theta$	$\sin^2 \theta$.
35	1 …… top,	$b-a=b \sin \theta$ or $b \cos \theta$	$b-a=b \sin^2 \theta$ or $b \cos^2 \theta$.
36	4 …… …..	$\cos c$	$\cos C$.
	6 …… …..	$\cos^2 \frac{c}{2}$	$\cos^2 \frac{C}{2}$.
	8 …… …..	$\sin^2 \frac{c}{2}$	$\sin^2 \frac{C}{2}$.
	10 …… …..	$\tan \frac{c}{2}$	$\tan \frac{C}{2}$.
	12 …… …..	$\cos \frac{c}{2}$	$\cos \frac{C}{2}$.
	4 …… bottom,	$2a$ vers θ	a vers θ.
37	12 …… top,	$\sec^3 \theta''$	$\sec^2 \theta''$.
	2 …… bottom,	$\tan^2 \theta$	$\tan^2 \theta'$.
38	1 …… top,	$\frac{c}{a''}=\frac{c}{a-b+c}$	$\frac{d}{a''}=\frac{d}{a-b+c}$.
	3 …… …..	$\frac{c}{a'''}$	$\frac{e}{a'''}$.
	8 …… bottom,	with	into.
39	9 …… top,	$A+\theta$	$A=\theta$.
40	4 …… …..	different	n different.
	5 …… bottom and 3 following	suppress the sign – before cos.	
43	5 …… bottom,	$(a^2+b^2)^{\frac{1}{8}}$	$(a^2+b^2)^{\frac{1}{6}}$.
53	9 …… top,	or a	or of a.

SYLLABUS

OF

TRIGONOMETRY.

CHAP. I.

ON THE REPRESENTATION OF LINES AND ANGLES.

1. LINES, considered with respect to magnitude only, may be represented by numbers or symbols, without any distinctive signs of affection attached to them.

2. The relation of lines which are geometrically[1] parallel, (that is, which make the same angle with the same straight line,) may be, in all cases, expressed by the signs + and −.[2]

3. Parallel lines, which are estimated in the same direction, (whether from right to left or from left to right, upwards or downwards, or the contrary,) are to be affected with the same algebraical signs, whether + or −.[3]

4. Parallel lines, which are estimated in different directions, are to be affected with different algebraical signs; those in one direction being affected with the sign +, and those in the opposite direction being affected with the sign −.[4]

---

[1] *Algebra*, Art. 559.

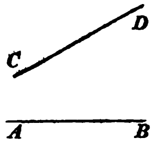
[2] *Algebra*, Art. 92.

[3] *Algebra*, Art. 92. 93.

[4] *Algebra*, Art. 92.

The same straight line AB, will have different algebraical signs (whether + and − or − and +), according as it is estimated in the direction AB or BA: and generally, whatever is asserted of the signs of affection of parallel lines, will apply to the signs of affection of lines, which are in the same straight line.

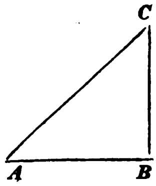
5. The primitive signs of affection, + and −, are not competent to express the relation of lines, (such as AB and CD,) to each other, which are not parallel.



The position of such lines (or their relation to each other, independently of their magnitudes) is dependent upon the angles which they make with each other.

6. Consideration of the different modes by which the angle BAC may be determined.[1]

Take *any* point C in one of the lines AC, and draw CB perpendicular to the other line AB: let the angle BAC be denoted by θ.



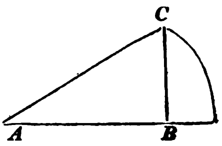
7. Definition of the sine of an angle.

The magnitude of the angle θ is determined by the ratio $\frac{CB}{AC}$. This ratio is called the *sine* of θ, and is written thus: sin θ.

---

[1] *Algebra*, Art. 465 and 549.

[2] This definition of the sine of an angle is different from that which is commonly given: if a circular arc be described from the centre A, with radius AC, then CB is commonly called the *sine* of θ to radius AC; in a similar manner, AB is commonly called the *cosine* of θ to radius AC.



8. **Definition of the cosine of an angle.**

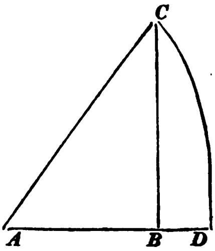
The magnitude of the angle θ is determined by the ratio $\frac{AB}{AC}$. This ratio is called the *cosine* of θ, and is written thus: $\cos\theta$.

9. The same angle may be determined by any of the ratios formed by two sides of the triangle ABC, such as

$$\frac{BC}{AB}, \quad \frac{AC}{AB}, \quad \frac{AC}{BC};$$

to each of which the specific names, *tangent* of θ, *secant* of θ, *cosecant* of θ; have been given: such ratios, however, are dependent upon those which have been designated *sine* and *cosine*, the properties of which will form the subject of the following Chapter.

10. With centre A and any radius AC, describe an arc CD of a circle, meeting AB produced on D.



The angle θ, which is *determined* by its sine $\frac{CB}{AC}$ or its cosine $\frac{AB}{AC}$, is both *determined* and *measured*[1] by the ratio $\frac{\text{arc } CD}{AC}$.

11. Assuming the preceding measure of an angle, we shall express angles either by their measures, or as aliquot parts of a right angle which is assumed as the primary angle, and with which all others are compared.

12. A right angle is supposed to be divided into 90 equal parts, each of which is called a *degree*.

---

[1] *Algebra*, Art. 666 and 667.

A degree is supposed to be divided into 60 equal parts, each of which is called a *minute*.

A minute is supposed to be divided into 60 equal parts, each of which is called a *second*.

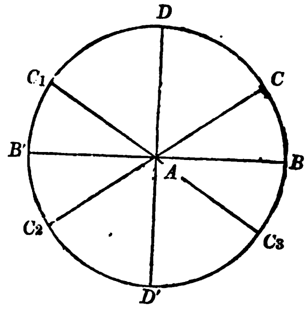
The same sexagesimal division may be continued indefinitely downwards, to *thirds*, *fourths*, &c.; but it is more usual, in modern works, to use the decimal rather than the sexagesimal divisions of the second.

The distinctive marks, $^\circ$, $'$, $''$, &c. are written above and to the right hand of the numbers or symbols, which severally denote the number of degrees, minutes, seconds, &c., which are contained in any given angle.

Thus, $18^\circ.17'.47''$ would designate an angle of 18 degrees, 17 minutes, and 47 seconds; $67^\circ.23'.50''.24'''$ would denote an angle of 67 degrees, 23 minutes, 50 seconds, and 24 thirds: the same angle, however, is more commonly denoted by $67^\circ..23'..50.4''$.

13. The ratio which the circumference of a circle bears to its diameter, is usually denoted by the symbol π: the ratio, therefore, which the circumference of a circle bears to its radius, must be denoted by 2π: and the ratio which a quadrantal arc of a circumference bears to the radius, must be denoted by $\frac{\pi}{2}$, which is therefore the measure of a right angle and which is always assumed to denote it.

14. Let a circle be described from the centre A, and with any radius AB; and let the two diameters BAB' and DAD' be drawn at right angles to each other.



The angles which AC can form with AB, (such as BAC or BAC_1), can never exceed π or 180^0, which is the extreme limit of an angle as considered in Geometry: the same limit applies to the angles BAC_2 and BAC_3, formed *below* the linc BAB'.

The angles formed *above* and *below* the line BAB', considered with respect to each other, would be affected with different algebraical signs: if the first be affected with the sign +, the second must be affected with the sign −; and conversely.

If the arc BC be affected with the sign +, the arc BC_3 being reckoned in the opposite direction to BC, must be affected with the sign −: the measures $\frac{BC}{AB}$ and $\frac{BC_1}{AB}$ of the superior angles BAC and BAC_1 are positive, whilst the measures $\frac{BC_2}{AB}$ and $\frac{BC_3}{AB}$ of the inferior angles are negative.

15. The estimation of angles by means of their measures has led to a great extension of the meaning of the term *angle*, as used in Geometry.

Suppose a point (C) and the radius (AC) along with it, to move round the circumference of a circle through the four quadrants BD, DB', $B'D'$, $D'B$, in the same direction. The corresponding arcs BC, BDC_1, $BDB'C_2$, $BDB'D'C_3$, being reckoned in the same direction, will be affected with the same algebraical sign, as well as the corresponding ratios

$$\frac{BC}{AB}, \frac{BDC_1}{AB}, \frac{BDB'C_2}{AB}, \frac{BDB'D'C_3}{AB}:$$

the angles generated likewise are

$$BAC, \ BAC_1, \ \pi + B'AC_2, \ \pi + B'AC_3,$$

which are measured by these ratios, and are affected with the same sign for all positions of the generating line AC.

We are thus led to consider angles which are greater than 180^0, and the position of the line AC_2 or AC_3 with respect to the primitive line AB, is equally determined by the positive angles $\pi + B'AC_2$ and $\pi + B'AC_3$, or by the negative angles BAC_2 and BAC_3.

If the angles BAC, $B'AC_1$, $B'AC_2$, BAC_3, be equal to each other and to θ, then AC makes the angle θ, AC_1 the angle $\pi - \theta$, AC_2 the angle $\pi + \theta$ or $-\pi + \theta$, and AC_3 the angle $2\pi - \theta$ or $-\theta$, with the primitive line AB.

16. The line AC, after revolving through four right angles, may continue its motion and pass through the same series of positions again: when it reaches the position AC for a second time, it will have revolved through an angle, the measure of which is

$$\frac{BDB'D'B + BC}{AB} = 2\pi + \theta:$$

if it reaches the position AC a third time, it will have revolved through an angle, whose measure is

$$\frac{2 \,.\, BDB'D'B + BC}{AB} = 4\pi + \theta:$$

and generally, if it reaches the position AC the n^{th} time, it will have revolved through an angle, whose measure is

$$\frac{(n-1)\, BDB'D'B + BC}{AB} = 2(n-1)\pi + \theta.$$

The following table will express the measures of the angles described from the primitive position AB, in successive revolutions of the generating line in the same direction, in order to reach the positions AC, AC_1, AC_2, AC_3.

Revolutions.	Position AC.	Position AC_1.	Position AC_2.	Position AC_3.
1^{st}	θ	$\pi-\theta$	$\pi+\theta$	$2\pi-\theta$
2^{d}	$2\pi+\theta$	$3\pi-\theta$	$3\pi+\theta$	$4\pi-\theta$
3^{d}	$4\pi+\theta$	$5\pi-\theta$	$5\pi+\theta$	$6\pi-\theta$
4^{th}	$6\pi+\theta$	$7\pi-\theta$	$7\pi+\theta$	$8\pi-\theta$
&c.	&c.	&c.	&c.	&c.
n^{th}	$2(n-1)\pi+\theta$	$(2n-1)\pi-\theta$	$(2n-1)\pi+\theta$	$2n\pi-\theta$

17. Considering angles or their measures as the means of determining the positions of lines with respect to a primitive line, we may add or subtract any multiple of 2π or of 360^0, to or from any expression for the measure of the angle described, without affecting the *position* which such an expression is required to indicate.

Thus, θ, $2\pi+\theta$, $4\pi+\theta$, .. $2(n-1)\pi+\theta$, may be used *indifferently*, inasmuch as they all of them equally indicate a line in the position AC, or algebraically parallel to it.

Again, $\pi-\theta$, $3\pi-\theta$, $5\pi-\theta$ .. $(2n-1)\pi-\theta$, may be used *indifferently*, inasmuch as they all of them equally indicate a line in the position AC_1 (or in the second quadrant), or algebraically parallel to it.

Again, $\pi+\theta$, $3\pi+\theta$, $5\pi+\theta$, .. $(2n-1)\pi+\theta$, may be used *indifferently*, inasmuch as they all of them equally indicate a line in the position AC_2 (or in the third quadrant), or algebraically parallel to it.

Again, $2\pi-\theta$, $4\pi-\theta$, $6\pi-\theta$, .. $2n\pi-\theta$, may be used *indifferently*, inasmuch as they all of them equally indicate a line in the position AC_3 (as in the fourth quadrant), or algebraically parallel to it.

In a similar manner, if we suppose the generating line to revolve in a negative direction, from B through D', B', D to B again, then we shall find

$$-2\pi+\theta,\ \ -4\pi+\theta,\ \ \&c.\ \ -2n\pi+\theta,$$

equivalent to θ, inasmuch as they will all of them indicate a line in the position AC, or algebraically parallel to it,

For the same reason,

$$-\pi-\theta,\ \ -3\pi-\theta,\ \ -5\pi-\theta,\ \dots\ -(2n-1)\pi-\theta,$$

are equivalent to $\pi-\theta$;

$$-\pi+\theta,\ \ -3\pi+\theta,\ \ -5\pi+\theta,\ \dots\ -(2n-1)\pi+\theta,$$

are equivalent to $\pi+\theta$;

$$\text{and}\ \ -\theta,\ \ -2\pi-\theta,\ \ -4\pi-\theta,\ \dots\ -2n\pi-\theta,$$

are equivalent to $2\pi-\theta$.

18. The same position or *angle* may correspond to an infinite number of measures of angles, differing from each other by 2π or its multiples: but only one position or angle can correspond to the same measure of an angle.

19. The angle CAD, denoted by $\frac{\pi}{2} - \theta$, is called the *complement* of the angle BAC or θ.

We call $\frac{\pi}{2} - \theta$, the *complement* of θ, *generally*, whether θ be greater or less then $\frac{\pi}{2}$.

20. The angle BAC_1, denoted by $\pi - \theta$, is called the *supplement* of θ.

We call $\pi - \theta$, the *supplement* of θ, *generally*, whether θ be greater or less than 180^0.

CHAP. II.

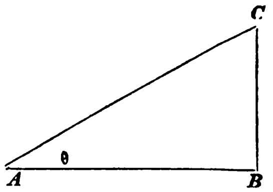
PROPERTIES OF THE SINES AND COSINES OF ANGLES.

1. Resumed consideration of the definitions of the *sine* and *cosine* of an angle. Art. 7. 8. Chap. I.

2. If $AC = a$, and the angle $BAC = \theta$, then we have

$$BC = AC \sin BAC = a \sin \theta,$$

$$AB = AC \cos BAC = a \cos \theta.$$



Or the side opposite to (the perpendicular) an acute angle of a right-angled triangle is equal to the product of the hypothenuse, and of the sine of the angle; and the side adjacent (the base) is equal to the product of the hypothenuse and of the cosine of the same angle.

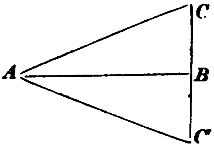
3. $\operatorname{Sin}^2 \theta + \cos^2 \theta = 1.$

This is the fundamental property which establishes the relation between the sine and cosine of an angle, and which enables us to express one of them in terms of the other.

Thus,
$$\sin \theta = \sqrt{(1 - \cos^2 \theta)},$$
$$\cos \theta = \sqrt{(1 - \sin^2 \theta)}.$$

4. $\operatorname{Sin} - \theta = - \sin \theta:$

$\cos - \theta = \quad \cos \theta.$



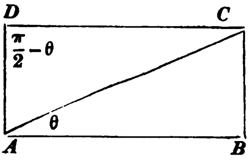
5. $\operatorname{Sin} 0 = 0: \qquad \cos 0 = 1.$

$\operatorname{Sin} \frac{\pi}{2} = 1: \qquad \cos \frac{\pi}{2} = 0.$

$\operatorname{Sin} \pi = 0: \qquad \cos \pi = -1.$

6. $\operatorname{Sin} \left(\frac{\pi}{2} - \theta\right) = \cos \theta:$

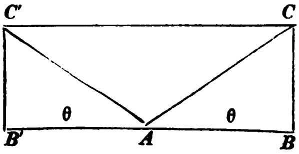
$\cos \left(\frac{\pi}{2} - \theta\right) = \sin \theta.$



Or the sine of an angle is equal to the cosine of its complement, and the cosine of an angle is equal to the sine of its complement.

7. $\operatorname{Sin} (\pi - \theta) = \sin \theta,$

$\cos (\pi - \theta) = - \cos \theta.$



Or the sine of an angle is equal to the sine of its supplement, and the cosine of an angle is equal to the cosine of its supplement, with its algebraical sign changed.

8. **Proposition.** To find expressions for the sines and cosines of the sum and difference of two angles, in terms of the sines and cosines of the component angles.

Let the component angles be denoted by θ and θ': then we shall find

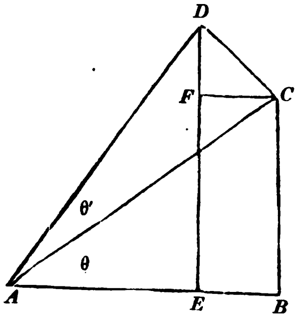
$$\sin(\theta+\theta') = \sin\theta\cos\theta' + \cos\theta\sin\theta' \ldots\ldots (1)$$

$$\cos(\theta+\theta') = \cos\theta\cos\theta' - \sin\theta\sin\theta' \ldots\ldots (2)$$

$$\sin(\theta-\theta') = \sin\theta\cos\theta' - \cos\theta\sin\theta' \ldots\ldots (3)$$

$$\cos(\theta-\theta') = \cos\theta\cos\theta' + \sin\theta\sin\theta' \ldots\ldots (4).$$

Demonstration of formulæ (1) and (2).



Let $BAC=\theta$, and $CAD=\theta'$; and therefore $BAD=\theta+\theta'$: let CB be drawn at right angles to AB, DC at right angles to AC, DE at right angles to AB, and CF at right angles to DE.

Then, from Art. 2, we get

$AC=AD\cos\theta'$: $\quad CD=AD\sin\theta'$:

$BC=AC\sin\theta=AD\sin\theta\cos\theta'$,

$AB=AC\cos\theta=AD\cos\theta\cos\theta'$,

$BE=CF=CD\sin CDF=CD\sin\theta$
$=AD\sin\theta\sin\theta'$;

$DF=CD\cos\theta=AD\cos\theta\sin\theta'$:

$DE=AD\sin(\theta+\theta')=BC+DF$
$=AD\sin\theta\cos\theta'+AD\cos\theta\sin\theta'$;

and, therefore, dividing both sides of the equation by AD, we get

$$\sin(\theta+\theta')=\sin\theta\cos\theta'+\cos\theta\sin\theta',$$

which is the first of the required formulæ.

Again, $\quad AE=AD\cos(\theta+\theta')=AB-CF$
$=AD\cos\theta\cos\theta'-AD\sin\theta\sin\theta'$;

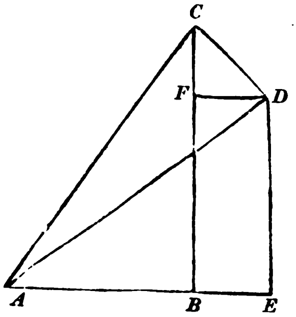
and dividing both sides of the equation by AD, we get

$$\cos(\theta+\theta')=\cos\theta\cos\theta'-\sin\theta\sin\theta',$$

which is the second of the required formulæ.

Demonstration of formulæ (3) and (4).

Let $BAC = \theta$, $CAD = \theta'$, and therefore $BAD = \theta - \theta'$: let CB be at right angles to AB, CD to AC, and DE to AE, and DF to BC.



Then, from Art. 2, we get

$$AC = AD \cos \theta' \ : \ CD = AD \sin \theta',$$
$$BC = AC \sin \theta = AD \sin \theta \cos \theta',$$
$$AB = AC \cos \theta = AD \cos \theta \cos \theta',$$
$$CF = DC \cos \theta = AD \cos \theta \sin \theta',$$
$$DF = DC \sin \theta = AD \sin \theta \sin \theta',$$

$$DE = AD \sin (\theta - \theta') = BC - CF = AD \sin \theta \cos \theta' - AD \cos \theta \sin \theta';$$

and dividing both sides of the equation by AD, we get

$$\sin (\theta - \theta') = \sin \theta \cos \theta' - \cos \theta \sin \theta',$$

which is the third of the required formulæ.

Again, $AE = AD \cos (\theta - \theta') = AB + DF$
$= AD \cos \theta \cos \theta' + AD \sin \theta \sin \theta';$

and dividing both sides of the equation by AD, we get

$$\cos (\theta - \theta') = \cos \theta \cos \theta' + \sin \theta \sin \theta',$$

which is the fourth of the required formulæ.

The three last formulæ may be deduced from the first, by means of Art. 4. 5. 6.

9. PROPOSITION. To express the sum or difference of the sines and of the cosines of two angles, in terms of the products of the sines or cosines of half the sum and half the difference of the angles,

$$\sin \theta + \sin \theta' = 2 \sin \left(\frac{\theta + \theta'}{2}\right) \cos \left(\frac{\theta - \theta'}{2}\right) \dots\dots\dots (1)$$

$$\sin \theta - \sin \theta' = 2 \cos \left(\frac{\theta + \theta'}{2}\right) \sin \left(\frac{\theta - \theta'}{2}\right) \dots\dots\dots (2)$$

$$\cos\theta + \cos\theta' = 2\cos\left(\frac{\theta+\theta'}{2}\right)\cos\left(\frac{\theta-\theta'}{2}\right)\ldots\ldots\ldots(3)$$

$$\cos\theta - \cos\theta' = -2\sin\left(\frac{\theta+\theta'}{2}\right)\sin\left(\frac{\theta-\theta'}{2}\right)\ldots\ldots(4).$$

The following formulæ result immediately from those just given:

$$\sin(\theta+\theta')+\sin(\theta-\theta')=2\sin\theta\cos\theta'\ldots\ldots\ldots(5)$$
$$\sin(\theta+\theta')-\sin(\theta-\theta')=2\cos\theta\sin\theta'\ldots\ldots\ldots(6)$$
$$\cos(\theta+\theta')+\cos(\theta-\theta')=2\cos\theta\cos\theta'\ldots\ldots\ldots(7)$$
$$\cos(\theta+\theta')-\cos(\theta-\theta')=-2\sin\theta\sin\theta'\ldots\ldots(8).$$

These formulæ (5), (6), (7), (8), are generally referred to under the following forms:

$$\sin(\theta+\theta')=2\sin\theta\cos\theta'-\sin(\theta-\theta')\ldots\ldots\ldots(9)$$
$$\sin(\theta+\theta')=2\cos\theta\sin\theta'+\sin(\theta-\theta')\ldots\ldots\ldots(10)$$
$$\cos(\theta+\theta')=2\cos\theta\cos\theta'-\cos(\theta-\theta')\ldots\ldots\ldots(11)$$
$$\cos(\theta+\theta')=-2\sin\theta\cos\theta'+\cos(\theta-\theta')\ldots\ldots(12).$$

10. The following propositions are immediate consequences of the preceding articles:

(1) $\operatorname{Sin}\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$.

(2) $\operatorname{Cos}\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = 2\cos^2\frac{\theta}{2} - 1 = 1 - 2\sin^2\frac{\theta}{2}$.

This proposition is frequently referred to under one of the following forms:

$$\cos\frac{\theta}{2} = \sqrt{\left(\frac{1+\cos\theta}{2}\right)} \text{ or } \cos\theta = \sqrt{\left(\frac{1+\cos 2\theta}{2}\right)},$$

$$\sin\frac{\theta}{2} = \sqrt{\left(\frac{1-\cos\theta}{2}\right)} \text{ or } \sin\theta = \sqrt{\left(\frac{1-\cos 2\theta}{2}\right)}.$$

(3) $\operatorname{Sin} 45^0 = \cos 45^0 = \frac{1}{\sqrt{2}}$.

(4) $\operatorname{Sin} 30^0 = \cos 60^0 = \frac{1}{2}$.

(5) $\text{Sin } 60^0 = \cos 30^0 = \frac{\sqrt{3}}{2}.$

(6) $\text{Sin } 120^0 = \cos 30^0 = \sin 60^0 = \frac{\sqrt{3}}{2}.$

(7) $\text{Cos } 120^0 = - \sin 30^0 = - \cos 60^0 = - \frac{1}{2}.$

(8) $\text{Sin } 135^0 = \cos 45^0 = \sin 45^0 = \frac{1}{\sqrt{2}}.$

(9) $\text{Cos } 135^0 = - \cos 45^0 = - \sin 45^0 = - \frac{1}{\sqrt{2}}.$

(10) $\text{Sin } 150^0 = \sin 30 = \cos 60 = \frac{1}{2}.$

(11) $\text{Cos } 150^0 = - \cos 30^0 = - \sin 60 = - \frac{\sqrt{3}}{2}.$

The preceding numerical values of the sines and cosines of 30^0, 45^0, 60^0, and of their *supplements*, are very frequently quoted, and the knowledge of them is extremely useful in the transformation and verification of goniometrical formulæ.

(12) $\text{Sin } \theta = \sin (\pm 2n\pi + \theta),$

$\cos \theta = \cos (\pm 2n\pi + \theta),$

where n is any whole number.

For θ and $\pm 2n\pi + \theta$ denote the same angle θ, though differing from each other as *measures*, by multiples of 2π, or of the measure of 360^0.

(13) $\text{Sin } \theta = \sin (\pi - \theta) = \sin \{\pm (2n + 1)\pi - \theta\},$

$- \cos \theta = \cos (\pi - \theta) = \cos \{\pm (2n + 1)\pi - \theta\},$

where n is any whole number.

(14) $- \sin \theta = \sin (- \theta) = \sin (\pm 2n\pi - \theta),$

$\cos \theta = \cos (- \theta) = \cos \pm (2n\pi - \theta),$

where n is any whole number.

(15) $$-\sin\theta = \sin(\pi+\theta) = \sin\left\{\pm(2n+1)\,\pi+\theta\right\},$$

$$-\cos\theta = \cos(\pi+\theta) = \cos\left\{\pm(2n+1)\,\pi+\theta\right\},$$

where n is any whole number.

(16) $$\text{Sin}\,\theta = \cos\left(\frac{\pi}{2}-\theta\right) = \cos\left\{\left(\frac{\pm 4n+1}{2}\right)\pi-\theta\right\},$$

$$\cos\theta = \sin\left(\frac{\pi}{2}-\theta\right) = \sin\left\{\left(\frac{\pm 4n+1}{2}\right)\pi-\theta\right\},$$

where n is any whole number.

(17) $$-\sin\theta = \cos\left(\frac{\pi}{2}+\theta\right) = \cos\left\{\left(\frac{\pm 4n+1}{2}\right)\pi+\theta\right\},$$

$$\cos\theta = \sin\left(\frac{\pi}{2}+\theta\right) = \sin\left\{\left(\frac{\pm 4n+1}{2}\right)\pi+\theta\right\},$$

where n is any whole number.

(18) $$-\sin\theta = \cos\left(\frac{3\pi}{2}-\theta\right) = \cos\left\{\left(\frac{\pm 4n+3}{2}\right)\pi-\theta\right\},$$

$$-\cos\theta = \sin\left(\frac{3\pi}{2}-\theta\right) = \sin\left\{\left(\frac{\pm 4n+3}{2}\right)\pi-\theta\right\},$$

where n is any whole number.

(19) $$\text{Sin}\,\theta = \cos\left(\frac{3\pi}{2}+\theta\right) = \cos\left\{\left(\frac{\pm 4n+3}{2}\right)\pi+\theta\right\},$$

$$-\cos\theta = \sin\left(\frac{3\pi}{2}+\theta\right) = \sin\left\{\left(\frac{\pm 4n+3}{2}\right)\pi+\theta\right\},$$

where n is any whole number.

The immense variety of equivalent forces of the sine and cosine of an angle, which are included in the eight Articles immediately preceding, results from considering angles as expressed by their measures and as generated by the successive revolutions, in the same plane, of one line about another which is assumed as the primitive line.

(20) $$\operatorname{Sin}\theta = \sin(60^0 + \theta) - \sin(60^0 - \theta),$$
$$= \cos(30 - \theta) - \cos(30 + \theta),$$
$$\cos\theta = \cos(60^0 + \theta) + \cos(60^0 - \theta),$$
$$= \sin(30^0 + \theta) + \sin(30^0 - \theta).$$

(21) $$\operatorname{Sin}\theta = \tfrac{1}{2}\sqrt{(1 + \sin 2\theta)} - \tfrac{1}{2}\sqrt{(1 - \sin 2\theta)},$$
$$\cos\theta = \tfrac{1}{2}\sqrt{(1 + \sin 2\theta)} + \tfrac{1}{2}\sqrt{(1 - \sin 2\theta)}.$$

In these formulæ, regard must be had to the algebraical sign of the square root, when θ is greater than 45^o.

(22) $$\operatorname{Sin} 3\theta = 3\sin\theta - 4(\sin\theta)^3,$$
$$\cos 3\theta = 4(\cos\theta)^3 - 3\cos\theta.$$

CHAP. III.

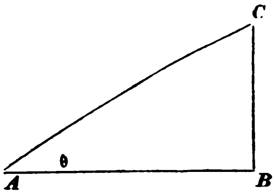
ON THE TANGENTS, COTANGENTS, SECANTS, COSECANTS AND VERSED SINES OF ANGLES.

1. THE ratio of the sine to the cosine of an angle is called its *tangent:* and the reciprocal ratio of the cosine to the sine of an angle is called its *cotangent.*

If θ be the angle, then

$$\text{tangent of } \theta \text{ or } \tan\theta = \frac{\sin\theta}{\cos\theta},$$

$$\text{cotangent of } \theta \text{ or } \cot\theta = \frac{\cos\theta}{\sin\theta}.$$



The tangent and cotangent of an angle *determine* the angle equally with the sine and cosine: if BAC be an angle, and if from any point C, CB be drawn at right angles to AB, then the ratio $\frac{BC}{AB}$ is the tangent and $\frac{AB}{BC}$ is the cotangent of the angle θ; and the *species* of the triangle is determined either by one or the other.

2. $\text{Tan}\,\theta = \frac{1}{\cot\theta}$.

3. $\text{Tan} - \theta = - \tan\theta$.

4. $\text{Tan}\,\theta = \cot\left(\frac{\pi}{2} - \theta\right)$:

$\cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$.

5. $\text{Tan}\,0^\circ = 0$: $\cot 0^\circ = \frac{1}{0} =$ infinity, or a quantity which exceeds any finite magnitude, and which is usually denoted by the symbol ∞.

6. $\text{Tan}\,30^\circ = \frac{1}{\sqrt{3}}$: $\cot 30^\circ = \sqrt{3}$.

7. $\text{Tan}\,45^\circ = 1$: $\cot 45^\circ = 1$.

8. $\text{Tan}\,60^\circ = \sqrt{3}$: $\cot 60^\circ = \frac{1}{\sqrt{3}}$.

9. $\text{Tan}\,\frac{\pi}{2} = \infty$: $\cot\frac{\pi}{2} = 0$.

10. $\text{Tan}\,\pi = 0$: $\cot\pi = -\infty$.

11. PROPOSITION. Given the tangents of two angles, to find the tangents of their sum and difference.

$$\tan(\theta+\theta') = \frac{\tan\theta+\tan\theta'}{1-\tan\theta\tan\theta'},$$

$$\tan(\theta-\theta') = \frac{\tan\theta-\tan\theta'}{1+\tan\theta\tan\theta'}.$$

12. $$\text{Tan}\,\theta = \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}.$$

13. $$\begin{aligned}\text{Tan}\,\theta &= -\tan(\pi-\theta),\\ \tan\theta &= \tan(\pi+\theta),\\ \cot\theta &= -\tan\left(\frac{\pi}{2}+\theta\right),\\ \cot\theta &= \tan\left(\frac{3\pi}{2}-\theta\right),\\ \cot\theta &= -\tan\left(\frac{3\pi}{2}+\theta\right).\end{aligned}$$

We have not put down the variations of these equivalent forms, which arise from adding to or subtracting from, the angles or their measures, any multiples of 2π. See Art. 10. Nos. 12 to 19. Chap. II.

14. $$\text{Tan}\,\theta = \sqrt{\left\{\frac{1-\cos 2\theta}{1+\cos 2\theta}\right\}}.$$

15. $\text{Tan}\,\theta = \cot\theta - 2\cot 2\theta.$

16. $2\tan 2\theta = \tan(45^\circ+\theta) - \tan(45^\circ-\theta).$

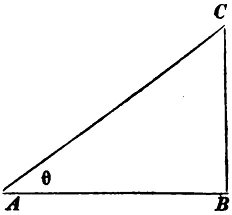
17. The reciprocal of the cosine of an angle is called its *secant*, and the reciprocal of the sine of an angle is called its *cosecant*.

If θ be an angle, then

$$\text{secant of } \theta \text{ or } \sec\theta = \frac{1}{\cos\theta};$$

$$\text{cosecant of } \theta \text{ or } \operatorname{cosec}\theta = \frac{1}{\sin\theta}.$$

If BAC be an angle, and if CB be drawn at right angles to AB, then the ratio $\frac{AC}{AB}$ is the *secant* and the ratio $\frac{AC}{BC}$ is the *cosecant*, of the angle BAC.



18. $\text{Sec}\ \theta = \sqrt{(1 + \tan^2 \theta)}$.

19. $\text{Sec}\ 0^0 = 1:$ $\qquad$ $\text{cosec}\ 0^0 = \infty$.

$\text{Sec}\ 30^0 = \frac{2}{\sqrt{3}}:$ $\qquad$ $\text{cosec}\ 30^0 = 2$.

$\text{Sec}\ 60^0 = 2:$ $\qquad$ $\text{cosec}\ 60^0 = \frac{2}{\sqrt{3}}$.

$\text{Sec}\ 45^0 = \sqrt{2}:$ $\qquad$ $\text{cosec}\ 45^0 = \sqrt{2}$.

$\text{Sec}\ \frac{\pi}{2} = \infty:$ $\qquad$ $\text{cosec}\ \frac{\pi}{2} = 1$.

$\text{Sec}\ \pi = -1:$ $\qquad$ $\text{cosec}\ \pi = \infty$.

$\text{Sec}\ \frac{3\pi}{2} = -\infty:$ $\qquad$ $\text{cosec}\ \frac{3\pi}{2} = -1$.

The values of the secants and cosecants of angles so readily follow from those of their cosines and sines, that it is not necessary to enter into a detailed examination of them.

20. The *versed sine* of an angle θ, written vers. θ, is used to express $1 - \cos\theta$.

21. $\text{Vers.}\ 0 = 1:$ $\qquad$ $\text{vers.}\ 30^0 = 1 - \frac{\sqrt{3}}{2}:$

$\text{vers.}\ 45^0 = 1 - \frac{1}{\sqrt{2}}:$ $\qquad$ $\text{vers.}\ 60^0 = \frac{1}{2}:$

$\text{vers.}\ \frac{\pi}{2} = 1:$ $\qquad$ $\text{vers.}\ \pi = 2:$ $\qquad$ $\text{vers.}\ \frac{3\pi}{2} = 1$.

The limits of the value of the versed sine are 1 and 2.

22. Vers. $\theta = 2 \sin^2 \frac{\theta}{2}$.

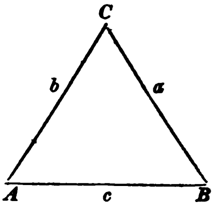
For further remarks upon the terms tangent and cotangent, secant and cosecant, and versed sine, their origin and uses, see *Algebra*, Art. 488, 489, 490, 491, 492 and 493.

CHAP. IV.

ON THE EXPRESSION OF THE COSINES AND SINES, AND OTHER GONIOMETRICAL FUNCTIONS OF THE ANGLES OF TRIANGLES IN TERMS OF THEIR SIDES.

FIRST ELEMENTS OF TRIGONOMETRY PROPERLY SO CALLED.

1. Let the three sides of a triangle (considered with respect to magnitude only) be represented by a, b, c; and let the angles opposite to them be denoted by A, B, C respectively.



2. $$\text{Cos } A = \frac{b^2 + c^2 - a^2}{2bc}: \quad \cos B = \frac{a^2 + c^2 - b^2}{2ac}:$$
$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

3. If $s = \frac{a+b+c}{2}$, then we have

$$1 + \cos A = 2 \cos^2 \frac{A}{2} = \frac{2s.(s-a)}{bc},$$

$$1 - \cos A = 2 \sin^2 \frac{A}{2} = \frac{2(s-b)(s-c)}{bc}.$$

4. If $S = \sqrt{\{s(s-a)(s-b)(s-c)\}}$, then

$$\sin A = \frac{2S}{bc}: \quad \sin B = \frac{2S}{ac}: \quad \sin C = \frac{2S}{ab}.$$

5. The area of the triangle

$$= \frac{bc \sin A}{2} = S = \sqrt{\{s(s-a)(s-b)(s-c)\}}.$$

Or the area of a triangle[1] is the square root of the continued product of half the sum of the sides and of its several excesses above each of the three sides.

6. $$\frac{\sin A}{\sin B} = \frac{a}{b}: \quad \frac{\sin A}{\sin C} = \frac{a}{c}: \quad \frac{\sin B}{\sin C} = \frac{b}{c}.$$

Or the sines of the angles of a triangle are proportional to the opposite sides.

7. $$\frac{a+b}{a-b} = \frac{\tan\left(\frac{A+B}{2}\right)}{\tan\left(\frac{A-B}{2}\right)}.$$

Or the sum of two sides of a triangle is to their difference, or the tangent of half the sum of the angles opposite to them is to the tangent of half their difference.

8. $c = \sqrt{(a^2 + b^2 - 2ab \cos C)}$.

This is an expression for a side of a triangle in terms of the remaining sides and of the cosine of the angle opposite to it: it will be found hereafter to admit of various modifications of form, chiefly through the aid of subsidiary angles.

9. If it be granted (as will be seen in the following Chapter,) that the *actual value* of an angle may be not only *determined* but *calculated* from its sine, cosine, tangent, &c., then the preceding propositions will enable us to determine the angles and sides of triangles from the requisite data.

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[1] *Algebra*, Art. 546. 547.

10. A triangle is determined from its three sides: the corresponding problem, therefore, is

"Given the three sides of a triangle, to find the three angles."

This may be done through the expressions for the cosines of the angles in terms of the three sides, as given in Art. 2; or through the corresponding expressions for their sines, as given in Art. 4.

11. A triangle is determined (in all cases but one) by two sides and an angle opposite to one of them: the corresponding problem is,

"Given two sides and an angle opposite to one of them, to find the remaining parts."

This may be effected by the proposition in Art. 6, which establishes the relation between the sides of a triangle and the sines of the angles opposite to them: thus,

$$\sin A = \frac{a}{b} \sin B = \frac{a}{c} \sin C,$$

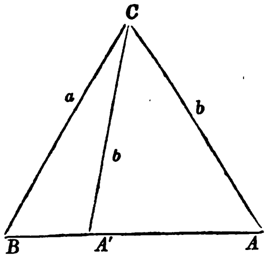
$$\sin B = \frac{b}{a} \sin A = \frac{b}{c} \sin C,$$

$$\sin C = \frac{c}{a} \sin A = \frac{c}{b} \sin B.$$

If the *given* angle be opposite to the *less* of the two sides, then there are two *different* triangles, which may possess the same data: consequently for this particular case the data are not sufficient to determine the triangle: under such circumstances, in deducing the value of the angle A from

$$\sin A = \frac{a}{b} \sin B,$$

it is *ambiguous*, whether we must make A less than 90^0 or greater than 90^0, the two values being necessarily supplemental to each other,



inasmuch as they have the same sine, and are limited to values less than 180^0.

No such ambiguity can present itself when the given angle is opposite to the greater side: for the greater angle is opposite to the greater side, and therefore A must be less than B, and therefore less than 90^0.

We thus get two sides and two angles, and therefore three angles; for $C = 180^0 - A - B$: the solution of the problem proposed in the next Article will enable us to determine the third side.

12. A triangle is determined by one side and two angles: and inasmuch as two angles determine the third, the corresponding problem may be stated thus:

"Given one side and all the angles of a triangle, to determine the remaining sides."

The solution is obtained from the formulæ derived from Art. 6.

$$a = \frac{b \, . \sin A}{\sin B} = \frac{c \, . \sin A}{\sin C}.$$

$$b = \frac{a \, . \sin B}{\sin A} = \frac{c \, . \sin B}{\sin C}.$$

$$c = \frac{a \, . \sin C}{\sin A} = \frac{b \, . \sin C}{\sin B}.$$

13. A triangle is determined by two sides, and the angle included by them: the corresponding problem may be stated thus:

"Given two sides and the included angle: to find the remaining parts."

In the first place, if a and b be the sides, and C the included angle, then we have

$$A + B = \pi - C; \text{ and from Art. 7,}$$

$$\tan\left(\frac{A-B}{2}\right) = \left(\frac{a-b}{a+b}\right) \tan\left(\frac{A+B}{2}\right) = \left(\frac{a-b}{a+b}\right) \cot\frac{C}{2}.$$

If $\frac{A+B}{2} = S$, and $\frac{A-B}{2} = D$, both of which are determined, we have $A = \frac{S+D}{2}$ and $B = \frac{S-D}{2}$, which are thus determined likewise.

The third side, when the three angles are known, may be determined by the formulæ in the last Article, or immediately (that is, without the intervention of the two angles A and B,) by the formula given in Art. 8.

14. The three angles of a triangle determine it in *species* only, and not in magnitude: the formulæ in Art. 6. will, in this case, determine the several ratios of its sides to each other.

CHAP. V.

ON THE CONSTRUCTION OF A CANON OF SINES AND COSINES, TANGENTS, AND COTANGENTS, SECANTS AND COSECANTS.

1. A *Canon* of sines and cosines is a table which contains their successive numerical values or their logarithms (increased by 10), corresponding to every degree and minute included between 1′ and 90°.

2. To calculate the numerical value of the sine and cosine of 1′.

If, in the formula (Art. 21. Chap. II.)

$$\sin\theta = \frac{1}{2}\sqrt{(1+\sin 2\theta)} - \frac{1}{2}\sqrt{(1-\sin 2\theta)},$$

we put for 2θ, successively 30°, $\frac{30^0}{2}$, $\frac{30^0}{2^2}$, $\frac{30^0}{2^3}$, as far as $\frac{30^0}{2^{11}}$, we shall find

$$\sin\frac{30^0}{2^{11}} = \sin\frac{225}{256} \times 1' = .000255625,$$

and therefore

$$\sin 1' = \frac{256}{225} \sin \frac{30^0}{2^{11}} \text{ (nearly[1])} = .0002908882;$$

from which we immediately obtain the value of the cosine of 1′.

3. To calculate the sines and cosines of all angles between 1′ and 30°.

$$\sin (n+1)\,\theta = 2 \cos \theta \sin n\theta - \sin (n-1)\,\theta,$$
$$\cos (n+1)\,\theta = 2 \cos \theta \cos n\theta - \cos (n-1)\,\theta.$$

If, in these formulæ, (deduced from Art. 9. Chap. II.), we make $\theta = 1'$, and substitute for n the numbers 1, 2, 3, &c. successively as far as 1799, we shall get the numerical values of the sines and cosines of all angles between 1′ and 30°.

4. To calculate the values of the sines and cosines of all angles between 30° and 45°.

$$\sin (30^0 + \theta) = \cos \theta - \sin (30^0 - \theta).$$

Make θ successively 1′, 2′, 3′, as far as 30°, and we get (the values of the sines and cosines of all angles as far as 30° being supposed to be known,) the sines of all angles between 30° and 60°: and, inasmuch as $\cos (30^0 + \theta) = \sin (60^0 - \theta)$, the sines of all angles from 60° to 45° downwards, will be the cosines of their complements from 30° to 45° upwards.

5. To calculate a canon of tangents and cotangents.

The tangents of any angle may be found by dividing the sine by the cosine, and the cotangent by dividing the cosine by the sine.

If we find the tangents as far as 45°, the tangents between 45° and 90°, or the cotangents between 45° and 0°, may be found from the formula, (Art. 16. Chap. III.)

$$\tan (45^0 + \theta) = \cot (45^0 - \theta) = 2 \tan 2\theta + \tan (45^0 - \theta).$$

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[1] There are methods of determining the value of the sine of 1′, *directly*, without assuming that the sines of small angles are in the same proportion with the angles themselves: they require, however, the approximate numerical solution of equations of three and five dimensions.

6. To calculate a canon of secants and cosecants.

The secants may be calculated immediately from a canon of sines and the cosecants from a canon of cosines.

Since $$\operatorname{cosec}\theta = \frac{\tan\frac{\theta}{2} + \cot\frac{\theta}{2}}{2},$$

$$\sec\theta = \frac{\tan\left(45^\circ + \frac{\theta}{2}\right) + \cot\left(45^\circ + \frac{\theta}{2}\right)}{2}.$$

the secants and cosecants of angles, which are *even* multiples of 1′, may be formed, by simple addition, from a canon of tangents and cotangents.

7. The formation of the Trigonometrical Canon may be effected by the methods just described: but more complete and expeditious methods have been employed for this purpose, the investigation of which would require the aid of a higher analysis than any which we are at present authorized to use.

8. The logarithms of the numerical values in the Trigonometrical Canon, have been calculated by methods which it is not required to explain in this Chapter[1]: their values have been registered in tables with the *same augment* of the number 10 throughout, in order to avoid the introduction and use of negative values.

9. In all methods of calculating the Trigonometrical Canon, the successive values are dependent upon those preceding them, and consequently an error in any one of them is transmitted through all those succeeding it.

10. In order to check the continued transmission of errors, as well as to verify the correctness of the successive

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[1] The sines and cosines of all angles whatsoever, as well as the tangents of all angles less than 45°, are less than 1, and consequently their logarithms (without their *augment*) are negative.

values, when no such errors exist, it is usual to interpose *stops* or values calculated by *independent* methods, at such intervals as may be considered necessary.

11. Such *stops* or *checks* are the numerical values corresponding to assigned angles, whose coincidence with the values corresponding to the same angles, determined by successive calculation, verifies the correctness of all the values in the preceding dependent series: or they are furnished by substituting two or more values already determined, in a goniometrical equation, which constitutes what is properly called a *formula of verification.*

12. The following are examples of numerical values, which furnish absolute and immediate *tests* of the correctness of the results obtained by *methods of continuation.*

(1) The values of the sines and cosines, tangents and cotangents, secants and cosecants, of 30^0, 45^0, 60^0, which have been given in Art. 10. 3. 4. 5. Chap. II.; Art. 6. 7. 8. and 19. Chap. III.

(2) The values of the sines and cosines of $\frac{30^0}{2}$, $\frac{30^0}{4}$, $\frac{30^0}{8}$, &c., and of $\frac{45^0}{2}$, $\frac{45^0}{4}$, $\frac{45^0}{8}$, &c., determined as in Art. 2. of this Chapter.

(3) $\text{Sin } 18^0 = \frac{\sqrt{5}-1}{4} = \cos 72^0$.

This value is obtained from the equation

$$\cos 54^0 = \sin 36^0,$$

$$\cos 18^0 \cos 36^0 - \sin 18^0 \sin 36^0 = \sin 36^0,$$

$$\cos 18^0 (1 - 2 \sin^2 18^0) - 2 \sin^2 18^0 \cos 18^0 = 2 \sin 18^0 \cos 18^0;$$

and dividing by $\cos 18^0$, we get

$$1 - 4 \sin^2 18^0 = 2 \sin 18^0:$$

from the solution of this equation, the preceding value of $\sin 18^0$ is obtained.

$$\cos 18^0 = \frac{\sqrt{10 + 2\sqrt{5}}}{4}.$$

$$\cos 36^0 = \sin 54^0 = \frac{\sqrt{5} + 1}{4}.$$

$$\sin 36^0 = \cos 54^0 = \frac{\sqrt{(10 - 2\sqrt{5})}}{4}.$$

(4) From the values of the sines and cosines 18^0 and 15^0, we get

$$\sin(18^0 - 15^0) = \sin 3^0 = \sin 18^0 \cos 15^0 - \cos 18^0 \sin 15^0,$$

(5) From the sine of 3^0, we can obtain $\sin 1^0$, by means of the approximate solution of the cubic equation, Art. 22. Chap. II.

$$\sin 3^0 = 3 \sin 1^0 - 4 \sin^3 1^0.$$

(6) The value of the sine of 1^0, and therefore of the cosine of 1^0, will enable us to calculate, by the formula of *continuation* in Art. 3. of this Chapter, the sines and cosines of 2^0, 3^0, 4^0, &c. as far as we choose to continue them.

We can thus make the sines and cosines of 1^0, 2^0, 3^0, &c. as far as 45^0, whether calculated by this method, or derived in whole or in part by any other means, so many successive *stops*, by which the values of the sines and cosines, calculated by a formula of continuation, may be verified.

If we find the sine and cosine of $30'$, from the sine of 1^0, we may by such means interpose a series of *stops* at intervals of $30'$, as far as 45^0; and similarly, we may interpose such *stops* at intervals of $15'$ or even of $5'$ (by the trisection of $15'$), if such should be considered necessary.

13. Formulæ of *verification* are equations between the sines, or the cosines, or the tangents of different angles, which, if satisfied by their values, as given in the Canon, or not, would verify the correctness of those values, or the contrary.

Of this kind are the equations given in Art. 20. and 21. Chap. II. and in Art. 16. Chap. III.

The following remarkable formulæ of verification are derived from the numerical values of the sine of 18^0, and of the cosine of 36^0, which are given in Art. 12. of this Chapter.

(1) $\text{Sin}\ \theta + \sin(72^0 + \theta) - \sin(72^0 - \theta)$
$= (\sin 36^0 + \theta) - \sin(36^0 - \theta)$.

(2) $\text{Sin}\ (90 - \theta) + \sin(18^0 + \theta) + \sin(18^0 - \theta)$
$= \sin(54^0 + \theta) + \sin(54^0 - \theta)$.

(3) $\text{Cos}\ \theta + \cos(72^0 + \theta) + \cos(72^0 - \theta)$
$= \cos(36^0 + \theta) + \cos(36^0 - \theta)$.

(4) $\cos(90 - \theta) + \cos(18^0 - \theta) - \cos(18^0 + \theta)$
$= \cos(54^0 - \theta) - \cos(54^0 + \theta)$.

CHAP. VI.

ON TABLES OF LOGARITHMS, AND THEIR APPLICATION TO THE CALCULATION OF GONIOMETRICAL AND OTHER FORMULÆ.

1. Definition of the logarithm of a number to an assumed *base*.[1]

2. Statement of what is meant by a *system* of logarithms.[2]

3. The logarithms which are exclusively used for the purposes of calculation, and which alone are extensively registered in *tables*, are calculated to a base 10, which is the basis or radix of our system of numeration.[3]

Such logarithms are called *tabular* logarithms, to distinguish them from the Napierian and other systems of logarithms, which are rarely registered in tables. See Chap. IX. Art. 5.

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[1] *Algebra*, Art. 678.
[2] *Algebra*, Art. 679.
[3] *Algebra*, Art. 681.

4. Statement of the general properties of logarithms.[1]

5. The arithmetical operations of multiplication and division of numbers, of raising their powers, and of extracting their roots, may be effected by the aid of a table of logarithms of all numbers, which are involved in or which result from, such operations.[2]

6. The integral part of a logarithm is called its *characteristic*, and the decimal part its *mantissa*.[3]

7. The characteristic of a logarithm is less by 1 than the number of integral places in the number.

If there are no integral places, the characteristic is

$$-1, \quad -2, \quad -3, \quad \text{or} \quad -n,$$

according as the first significant digit is in the first, second, third, or n^{th} place of decimals.

In order to denote *negative* characteristics, the sign $-$ is placed *above* the characteristic, for the purpose of shewing that it applies to the characteristic only, and not to the *mantissa* which follows.

8. The *mantissa*, or the decimal part of the logarithms of all numbers, which differ from each other in the place of units *only*, is the same. In other words, the logarithms of N, $10^n \times N$, $\frac{N}{10^n}$, differ from each other in their characteristics only.[4]

9. Inasmuch as the characteristic is immediately known from inspection of the number, it is always omitted in tables of the logarithms of *numbers*, which therefore contain *mantissæ* only.[5]

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[1] *Algebra*, Art. 685. and 686.

[2] *Algebra*, Art. 687.

[3] *Algebra*, Art. 689.

[4] *Algebra*, Art. 628.

[5] *Algebra*, Art. 690.

Statement of the great abbreviation of the size (not extent) of the tables of logarithms, which results from the omission of the characteristic.

A similar abbreviation would not be practicable in tables of logarithms which were calculated to any base, which differed from the base of our system of numeration.

10. The ordinary tables of logarithms contain the *mantissæ* of the logarithms of all numbers, from 1 to 100000, calculated to 7 places of decimals.[1]

11. They are also accompanied by a column containing the *differences* of successive *mantissæ*, considered as whole numbers: and also with an attached column, containing the *proportional parts* corresponding to every successive value of the *difference*.[2]

12. To find the logarithm of a number which consists of more than 5 digits, and which is therefore beyond the extent of the tables.

Its characteristic is known from inspection of the number; its *mantissa* may be found by the following rule:

"Find (from the tables) the *mantissa* of the number formed by the first 5 digits: from the column of differences, marked (D), find the *difference* corresponding to it: from the column (marked Pro.) of proportional parts, take the number corresponding to the 6th and $\frac{1}{10}$th of the number corresponding to the 7th digit, if any such digit exist: add this number or these numbers to the *mantissa* of the number formed by the first five digits, and we shall obtain the *mantissa* of the number of 6 or 7 places required."

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[1] Such tables are those of Sherwin, Hutton, Babbage, &c.; there are smalle tables of logarithms, which are calculated to 3 or 4 places of decimals only.

[2] *Algebra*, 707. 708, 710. In order to understand these Articles, the studen should be acquainted with the ordinary logarithmic series. See Chap. IX. Art. 6. 7. 8. &c.

13. To find the number corresponding to a logarithm, whose *mantissa* is not to be found in the tables; the following is the rule:

"Find (δ) the difference between the given *mantissa* and the next superior *mantissa* in the tables: find (d) the difference of the *mantissæ* of the two successive nearest numbers, as given in the column of differences (D): subjoin to the digits of the inferior number, the decimal arising from the fraction $\frac{\delta}{d}$, and the resulting number is the number required; the position of the decimal point being determined by the *characteristic* of the given logarithm."

14. In the logarithmic canon of sines, cosines, tangents, cotangents, secants, cosecants, versed sines, the logarithms, as given in the tables, are all increased by a common *augment* 10, in order to avoid the introduction of negative logarithms. See Art. 8. Chap. v.

15. In the determination of angles from the logarithms of their sines, cosines, &c. and in the logarithmic computation of formulæ in which they are involved, the *tabular* (or *augmented*) logarithms are always understood, and a most careful attention must be paid to this circumstance in the application of logarithmic computation to such formulæ.

16. Formulæ are said to be in *a state adapted to logarithmic computation*, when they consist of factors (in the numerator or denominator, or both), which require the operations of addition and subtraction *alone* for their computation, preparatory to the application of logarithms.

This phrase, however, is not used in a very strict sense, and formulæ might be said to be more or less adapted to logarithmic computation, according as they admitted of a more or less rapid arithmetical computation of their component factors.

17. In the adaptation of goniometrical formulæ to logarithmic computation, we must write in the logarithmic equations which result,

(log sin $A - 10$) instead of log sin A,
(log cos $A - 10$) instead of log cos A,
(log tan $A - 10$) instead of log tan A,
(log cot $A - 10$) instead of log cot A,
(log sec $A - 10$) instead of log sec A,
(log cosec $A - 10$) instead of log sec A,
(log vers $A - 10$) instead of log vers A.

18. Examples of the adaptation and reduction of formulæ to logarithmic computation.

(1) $x = a \sin A$.

$$\log x = \log a + \log \sin A - 10.$$

(2) $x = a \sin^2 A$.

$$\log x = \log a + 2 \log \sin A - 20.$$

(3) $x = a \cos A \sin B$.

$$\log x = \log a + \log \cos A + \log \sin B - 20.$$

(4) $a = \frac{b \sin A}{\sin B}$, Art. 12. Chap. IV.

$$\log a = \log b + \log \sin A - \log \sin B.$$

(5) $\text{Sin } A = \frac{a \sin B}{b}$, Art. 11. Chap. IV.

$$\log \sin A = \log \sin B + \log a - \log b.$$

(6) $\text{Cos } \frac{A}{2} = \sqrt{\left\{\frac{s.(s-a)}{bc}\right\}}$, Art. 3. Chap. IV.

$$\log \cos \frac{A}{2} = 10 + \frac{1}{2}\{\log s + \log (s-a)\} - \frac{1}{2}(\log b + \log c).$$

(7) $\operatorname{Sin}\frac{A}{2} = \sqrt{\left\{\frac{(s-b)(s-c)}{bc}\right\}}$, Art. 3. Chap. IV.

$$\log\sin\frac{A}{2} = 10 + \frac{1}{2}\{\log(s-b) + \log(s-c)\} - \frac{1}{2}(\log b + \log c).$$

(8) $\operatorname{Tan}\frac{A}{2} = \sqrt{\left\{\frac{(s-b)(s-c)}{s(s-a)}\right\}}$.

$$\log\tan\frac{A}{2} = 10 + \frac{1}{2}\{\log(s-b) + \log(s-c)\} - \frac{1}{2}\{\log s + \log(s-a)\}.$$

(9) $\operatorname{Sin} A = \frac{2\sqrt{\{s(s-a)(s-b)(s-c)\}}}{bc}$,
Art. 4. Chap. IV.

$$\log\sin A = 10 + \log 2 + \frac{1}{2}\{\log s + \log(s-a) + \log(s-b) + \log(s-c)\} - (\log b + \log c).$$

The formula $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$, is not properly adapted to logarithmic computation, inasmuch as one of its factors $(b^2 + c^2 - a^2)$, cannot be found without forming the squares of a^2, b^2 and c^3, by an arithmetical process.

(10) $\operatorname{Tan}\left(\frac{A-B}{2}\right) = \frac{(a-b)}{(a+b)}\tan\left(\frac{A+B}{2}\right)$,
Art. 8. Chap. V.

$$\log\tan\left(\frac{A-B}{2}\right) = \log\tan\left(\frac{A+B}{2}\right) + \log(a-b) - \log(a+b).$$

(11) $\operatorname{Tan}\left(\frac{A-B}{2}\right) = \tan(\theta - 45^0)\tan\left(\frac{A+B}{2}\right)$,

$$\log\tan\left(\frac{A+B}{2}\right) = \log\tan(\theta - 45^0) + \log\tan\left(\frac{A+B}{2}\right) - 10.$$

CHAP. VII.

ON THE USE AND APPLICATION OF SUBSIDIARY ANGLES.

1. A *Subsidiary* angle is one, whose sine, cosine, tangent, &c. does not exist in the primitive formula, but which is merely introduced for the purpose of facilitating its computation, whether by logarithms or otherwise.

2. The following are examples of the use and application of subsidiary angles.

(1) Binomial expressions of the form $a+b$ and $a-b$, where a and b are either simple numbers or compound expressions, immediately adaptable to logarithmic computation.

In the first, $\left\{(a+b)=a\left(1+\frac{b}{a}\right)\right\}$, make $\tan^2\theta=\frac{b}{a}$, where θ is the subsidiary angle, whose value is to be found from the tables:[*] then $a+b=a\,.\sec^2\theta=\frac{a}{\cos\theta}$.[†]

Whatever be the value of a and b (if possible and positive) the equation $\tan^2\theta=\frac{b}{a}$ may be satisfied.

In the second, $\left\{(a-b)=a\left(1-\frac{b}{a}\right)\right\}$, make $\frac{b}{a}=\cos^2\theta$ then $a-b=a\sin^2\theta$: or make $\frac{b}{a}=\sin^2\theta$, which gives

$$a-b=a\cos^2\theta.$$

The equations $\frac{b}{a}=\cos^2\theta$ or $\frac{b}{a}=\sin^3\theta$,' may be always satisfied, if a and b be positive and possible, and if a be greater than b; if a be less than b, then $a-b$ must be put under the form $-(b-a)$, and making $\frac{a}{b} = \cos^2\theta$ or $\frac{a}{b} = \sin^2\theta$, we shall get $b - a = b\sin\theta$ or $b\cos\theta$: the sign of $b-a$ thus calculated, must be subsequently changed from $+$ to $-$.

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[*] $\text{Log}\tan\theta=10+\frac{1}{2}(\log b-\log a)$.

[†] $\text{Log}\,(a+b)=\log a+2\log\sec\theta-20$
$=\log a+20-2\log\cos\theta$.

If a be expressed by $\alpha\sin^2 A$, where α is a given coefficient and A a given angle, we may make $b = \alpha\sin^2\theta$, when we get

$$a - b = \alpha(\sin^2 A - \sin^2\theta) = \alpha\sin(A+\theta)\sin(A-\theta).$$

This mode of introducing a subsidiary angle is not unfrequently used.

(2) Expressions such as $\frac{a-b}{a+b}$, make $\tan\theta = \frac{a}{b}$, which gives us $\tan(\theta - 45^0) = \frac{a-b}{a+b}$: or make

$$\cot\theta = \frac{a}{b} \text{ or } \tan\theta = \frac{b}{a},$$

which gives $\tan(45^0 - \theta) = \frac{a-b}{a+b}$.

Thus, the formula, Art. 7. Chap. v.

$$\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\tan\left(\frac{A+B}{2}\right),$$

becomes $\tan\left(\frac{A-B}{2}\right) = \tan(\theta - 45^0)\tan\left(\frac{A+B}{2}\right)$.

(3) Expressions such as $\sqrt{(a^2 - b^2)}$ and $\sqrt{(a^2 + b^2)}$.

In the first, if we make $\frac{b}{a} = \sin\theta$, we get

$$\sqrt{(a^2 - b^2)} = a\cos\theta:$$

and if we make $\frac{b}{a} = \cos\theta$, we get $\sqrt{(a^2 - b^2)} = a\sin\theta$.

In the second, if we make $\frac{b}{a} = \tan\theta$, we get

$$\sqrt{(a^2 + b^2)} = a\sec\theta = \frac{a}{\cos\theta}:$$

and if we make $\cot\theta = \frac{b}{a}$, we get

$$\sqrt{}(a^2 + b^2) = a \text{ cosec } \theta = \frac{a}{\sin\theta}.$$

The expression, Art. 8. Chap. v.

$$c = \sqrt{}(a^2 + b^2 - 2ab \cos c),$$

affords a good example of such transformations.

If we make $\sin^2\theta = \frac{4ab \cos^2 \frac{c}{2}}{(a+b)^2}$, we get

$$c = (a+b) \,.\, \cos\theta.$$

If we make $\tan^2\theta = \frac{4ab \,.\, \sin^2 \frac{c}{2}}{(a-b)^2}$, we get

$$c = (a-b) \sec\theta = \frac{a-b}{\cos\theta}.$$

If we make $\tan\theta = \frac{a+b}{a-b} \,.\, \tan\frac{c}{2}$, we get

$$c = (a-b) \cos\frac{c}{2} \sec\theta = (a-b)\frac{\cos\frac{c}{2}}{\cos\theta}.$$

(4) Expressions such as $a \pm \sqrt{}(a^2 - b^2)$ and $a \pm \sqrt{}(a^2 + b^2$

In the first, if we make $\frac{b}{a} = \sin\theta$, we get

$$a + \sqrt{}(a-{}^2b^2) = a\,(1 + \cos\theta) = 2a \cos^2\frac{\theta}{2} \text{.........(1)},$$

and $a - \sqrt{}(a^2-b^2) = a\,(1 - \cos\theta) = 2a \sin^2\frac{\theta}{2}$.........(2),

$$= 2a \text{ vers } \theta.$$

In the second, if we make $\cot\theta = \frac{a}{b}$, we get

$$a + \sqrt{}(a^2 + b^2) = b\,(\cot\theta + \text{cosec }\theta)$$

$$= b \,.\, \left(\frac{1 + \cos\theta}{\sin\theta}\right) = b \cot\frac{\theta}{2}.$$

$$a - \sqrt{(a^2 + b^2)} = b\,.\,(\cot\theta - \operatorname{cosec}\theta) = -\,b\,.\left(\frac{1 - \cos\theta}{\sin\theta}\right)$$

$$= -\,b\tan\frac{\theta}{2}\,.$$

Formulæ such as these, are furnished by the roots of the equations

$$\left.\begin{aligned} x^2 - px + q &= 0 \\ x^2 - px - q &= 0 \end{aligned}\right\};$$

which are thus reducible, when possible, to logarithmic calculation.

(5) To calculate a series of terms connected by the sign +, such as

$$a + b + c + d + \&c.$$

$$a + b = a\sec^2\theta = a, \text{ if } \tan^2\theta = \frac{b}{a},$$

$$a + b + c = a' + c = a'\sec^2\theta' = a\sec^2\theta\sec^2\theta' = a'',$$

$$\text{if } \tan^2\theta' = \frac{c}{a'} = \frac{c}{a+b}:$$

$$a + b + c + d = a'' + d = a''\sec^2\theta'' = a\sec^2\theta\sec^2\theta'\sec^3\theta'' = a''',$$

$$\text{if } \tan^2\theta'' = \frac{d}{a''} = \frac{d}{a+b+c}:$$

and so on for any number of terms of the series.

(6) To calculate a series of converging terms, such as

$$a - b + c - d + \&c.$$

connected alternately by signs + and −,

$$a - b = a\cos^2\theta = a', \text{ if } \frac{b}{a} = \sin^2\theta:$$

$$a - b + c = a'\sec^2\theta' = \frac{a\cos^2\theta}{\cos^2\theta'} = a'', \text{ if } \frac{c}{a'} = \frac{c}{a-b} = \tan^2\theta:$$

$$a - b + c - d = a''\cos^2\theta'' = \frac{a\cos^2\theta\cos^2\theta''}{\cos^2\theta'} = a''',$$

$$\text{if } \frac{c}{a''} = \frac{c}{a-b+c} = \sin^2\theta'' :$$

$$a-b+c-d+e = a''' \sec^2\theta''' = \frac{a\cos^2\theta\cos^2\theta''}{\cos^2\theta'\cos^2\theta'''}.$$

$$\text{if } \frac{c}{a'''} = \frac{e}{a-b+c-d} = \tan^2\theta'' :$$

(7) To calculate a continued product, such as

$$(a'+b')\,(a''+b'')\,(a'''+b''')\,\ldots\ldots \text{ or}$$

$$(a'-b')\,(a''-b'')\,(a'''-b''')\,\ldots\ldots$$

In the first, make $\tan^2\theta' = \frac{b'}{a'}$, $\tan^2\theta'' = \frac{b''}{a''}$, $\tan^2\theta''' = \frac{b'''}{a'''}$, &c. and we get

$$(a'+b')\,(a''+b'')\,(a'''+b''')\ldots = \frac{a'a''a'''\ldots\ldots}{\cos^2\theta'\cos^2\theta''\cos^2\theta'''\ldots}.$$

In the second, make

$$\sin^2\theta' = \frac{b'}{a'},\quad \sin^2\theta'' = \frac{b''}{a''},\quad \sin^2\theta''' = \frac{b'''}{a'''},\quad \&c.$$

and we get

$$(a'-b')\,(a''-b'')\,(a'''-b''')\ldots = a'a''a'''\ldots\cos^2\theta'\cos^2\theta''\cos^2\theta'''.$$

3. The following conversions of expressions, such as

$$a\sin A \pm b\cos A,\quad a\cos A \pm b\sin A,$$

with others of the form

$$a\sin(A\pm\theta),\quad a\cos(A\pm\theta),$$

by means of the subsidiary angle θ, are frequently of very great importance.

(1) $a\sin A \pm b\cos A = a\sin(A\pm\theta)$, if we make

$$a = \sqrt{(a^2+b^2)},\quad a = a\cos\theta \text{ and } b = a\sin\theta.$$

(2) $a\cos A \pm b\sin A = a\cos(A\mp\theta)$, if we make

$$a = \sqrt{(a^2+b^2)},\quad a = a\cos\theta,\quad b = a\sin\theta.$$

CHAP. VIII.

ON DEMOIVRE'S FORMULA, AND THE EXPRESSION BY MEANS OF IT OF THE ROOTS OF 1.

1. $(\text{Cos}\ \theta \pm \sqrt{-1}\ \sin\theta)^n = \cos n\theta \pm \sqrt{-1}\ \sin n\theta$, for all values of n: this theorem is usually called Demoivre's Theorem.

The steps of its demonstration are as follows:

$$(\cos A + \sqrt{-1}\ \sin A)\ (\cos\theta + \sqrt{-1}\ \sin\theta)$$
$$= \cos(A+\theta) + \sqrt{-1}\ .\ \sin(A+\theta).$$

Make $A+\theta$, and we get

$$(\cos\theta + \sqrt{-1}\ \sin\theta)^2 = \cos 2\theta + \sqrt{-1}\ \sin 2\theta).$$

Make $A=2\theta$, and we get

$$(\cos 2\theta + \sqrt{-1}\ .\ \sin 2\theta)\ (\cos\theta + \sqrt{-1}\ \sin\theta)$$
$$= (\cos\theta\ \sqrt{-1}\ \sin\theta)^3 = \cos 3\theta + \sqrt{-1}\ .\ \sin 3\theta.$$

The law of formation of the formula being thus indicated, assume

$$(\cos\theta + \sqrt{-1}\ .\ \sin\theta)^{n-1} = \cos(n-1)\ \theta + \sqrt{-1}\ \sin(n-1)\ \theta,$$

and make $A=(n-1)\ \theta$: we thus get

$$\{\cos(n-1)\ \theta + \sqrt{-1}\ \sin(n-1)\ \theta\}\ (\cos\theta + \sqrt{-1}\ \sin\theta)$$
$$= (\cos\theta + \sqrt{-1}\ \sin\theta)^n = \cos n\theta + \sqrt{-1}\ .\ \sin n\theta.$$

It thus appears, that if the formula be true for

$$\cos(n-1)\theta + \sqrt{-1}\ \sin(n-1)\theta,$$

it is necessarily true for

$$\cos n\theta + \sqrt{-1}\ \sin n\theta:$$

and as it has been shewn to be true when n is 2 or 3, it is therefore true when n is 4, 5, 6, &c. and therefore for any whole number whatsoever.

The principle of equivalent forms (*Algebra*, Art. 132.) authorizes us to conclude that this formula, which is true when n is *general in form*, though *specific* in value, is true likewise when n is *general* in value as well as in form.

2. If in the equation

$$(\cos\theta \pm \sqrt{-1}\sin\theta)^n = \cos n\theta \pm \sqrt{-1}\,.\sin n\theta,$$

the values of $n\theta$ are to be determined from the value of its cosine or sine, then there are different values of θ which give different values of

$$\cos\theta + \sqrt{-1}\sin\theta,$$

and which equally satisfy the equation.

For the angles $n\theta$, $2\pi + n\theta$, $4\pi + n\theta, \ldots 2r\pi + n\theta$, have the same sines and cosines, whatever be the *integral* value of r.

This series of angles, *divided by* n, produces the series

$$\theta,\ \frac{2\pi}{n}+\theta,\ \frac{4\pi}{n}+\theta, \ldots \frac{2(n-1)\pi}{n}+\theta,\ 2\pi+\theta,\ 2\pi+\frac{2\pi}{n}+\theta, \ldots,$$

the cosines and sines of the n first of which, one or both of them, are different from each other: the cosine and sine of

$$2\pi+\theta,\ 2\pi+\frac{2\pi}{n}+\theta,\ 2\pi+\frac{4\pi}{n}+\theta,\ \&c.$$

are respectively identical with the cosine and sine of

$$\theta,\ \frac{2\pi}{n}+\theta,\ \frac{4\pi}{n}+\theta,\ \&c.$$

and therefore no *new* values of $\cos\theta + \sqrt{-1}\sin\theta$, are produced by taking terms in this series, beyond the n first of them.

Any one, therefore, of the n values,

(1) $-\cos\theta + \sqrt{-1}\sin\theta$.

(2) $-\cos\left(\frac{2\pi}{n}+\theta\right) + \sqrt{-1}\sin\left(\frac{2\pi}{n}+\theta\right)$.

(3) $-\cos\left(\frac{4\pi}{n}+\theta\right) + \sqrt{-1}\sin\left(\frac{4\pi}{n}+\theta\right)$.

. . . .

($n-1$) $-\cos\left(\frac{2(n-2)\pi}{n}+\theta\right) + \sqrt{-1}\sin\left(\frac{2(n-2)\pi}{n}+\theta\right)$,

(n) .. $\cos\left(\frac{2(n-1)\pi}{n}+\theta\right) + \sqrt{-1}\sin\left(\frac{2(n-1)\pi}{n}+\theta\right)$;

which are different from each other, will satisfy, when raised to the n^{th} power, the *same* equation

$$\cos n\theta + \sqrt{-1} \sin n\theta,$$

where $n\theta$ is to be determined from the value of its sine or cosine.

Since

$$\cos\left(\frac{2(n-r)\pi}{n} + \theta\right) + \sqrt{-1} \sin\left(\frac{2(n-r)\pi}{n} + \theta\right)$$
$$= \cos\left(\frac{2r\pi}{n} - \theta\right) - \sqrt{-1} \sin\left(\frac{2r\pi}{n} - \theta\right);$$

the whole preceding series of n different values may be reduced to equivalent forms, which do not involve any angle exceeding 180^0.

3. To express the n roots of 1; or the n values of x in the equation

$$x^n - 1 = 0.$$

If, in the equation

$$\left\{\cos\left(\frac{2r\pi}{n} + \theta\right) + \sqrt{-1} \sin\left(\frac{2r\pi}{n} + \theta\right)\right\}^n = \cos n\theta + \sqrt{-1} \sin n\theta,$$

we make $\theta = 0$, and therefore $\cos n\theta = 1$ and $\sin n\theta = 0$, we get

$$\left(\cos \frac{2r\pi}{n} + \sqrt{-1} \sin \frac{2r\pi}{n}\right)^n = 1;$$

$$\text{and therefore } \cos \frac{2r\pi}{n} + \sqrt{-1} \sin \frac{2r\pi}{n} = (1)^{\frac{1}{n}};$$

in which r may be made $0, 1, 2, \ldots (n-1) \ldots$ successively, giving n results and *no more*, which are *different* from each other.

4. To express the n roots of -1; or the n values of x in the equation

$$x^n + 1 = 0.$$

If, in the equation

$$\left\{\cos\left(\frac{2r\pi}{n} + \theta\right) + \sqrt{-1} \sin\left(\frac{2r\pi}{n} + \theta\right)\right\}^n = \cos n\theta + \sqrt{-1} \sin n\theta,$$

we make $n\theta = \pi$, and therefore $\cos n\theta = -1$ and $\sin n\theta = 0$, we get

$$\left\{\cos\left(\frac{(2r+1)\pi}{n}\right) + \sqrt{-1} \sin\left(\frac{(2r+1)\pi}{n}\right)\right\}^n = -1;$$

and therefore $\cos \frac{(2r+1)\pi}{n} + \sqrt{-1} \sin \frac{(2r+1)\pi}{n} = (-1)^{\frac{1}{n}}$;

in which r may be made $0, 1, 2, \ldots (n-1) \ldots$ successively, giving n results and *no more*, which are *different* from each other.

5. To express the complete values of $(a)^n$ and $(-a)^n$, considered as derived from a and $-a$, for all values of n.

Assume ρ to represent the arithmetical value of $(a)^n$ or $(-a)^n$, which is independent of any sign of affection, and which is the same therefore for both.

Then, $(a)^n = (1)^n \rho = (\cos 2nr\pi + \sqrt{-1} \sin 2nr\pi)\, \rho$,

$$(-a)^n = (-1)^n \rho = \{\cos (2r+1)\, n\pi + \sqrt{-1} \sin (2r+1)\, n\pi\}\rho,$$

where r is any number in the series $0, 1, 2, 3, \ldots$

If n be any whole number, $(a)^n = \rho$.

If n be an even whole number, $(-a)^n = \rho$.

If n be an odd whole number, $(-a)^n = -\rho$.

If n be a rational fraction, having a denominator p: then the values of r may be limited to the terms of the series $0, 1, 2 \ldots . p-1$: beyond these limits, whether upwards or downwards, the same values recur.

If n be an irrational quantity, every *different* value of r will give a *different* value of $(a)^n$ or $(-a)^n$; and such values therefore never recur.

6. To express the complete values of

$$(a + b\sqrt{-1})^n \text{ and of } (a - b\sqrt{-1})^n,$$

considered as derived respectively from

$$a + b\sqrt{-1} \text{ and } a - b\sqrt{-1}.$$

Assume $\sqrt{(a^2 + b^2)}$ as the arithmetic value both of $a + b\sqrt{-1}$, and of $a - b\sqrt{-1}$;* and therefore $\rho = (a^2 + b^2)^{\frac{n}{2}}$, as the arithmeitc value both of $(a + b\sqrt{-1})^n$ and of $(a - b\sqrt{-1})^n$.

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* This assumption will be justified in Chap. x.

Suppose $\frac{a}{\sqrt{(a^2+b^2)}}=\cos\theta=\cos(2r\pi+\theta)$, and therefore

$$\frac{b}{\sqrt{(a^2+b^2)}}=\sin\theta=\sin(2r\pi+\theta);$$

where r is any whole number in the series... -3, -2, -1, 0, 1, 2, 3, ...

$$(a+b\sqrt{-1})^n=\rho\,\{\cos(2r\pi+\theta)+\sqrt{-1}\sin(2r\pi+\theta)\}^n,$$

$$=\rho\,\{\cos n\,(2r\pi+\theta)+\sqrt{-1}\sin n\,(2r\pi+\theta)\},$$

and $(a-b\sqrt{-1})^n=\rho\,\{\cos n\,(2r\pi+\theta)-\sqrt{-1}\sin n\,(2r\pi+\theta)\}$.

If we suppose $b=0$, and therefore $\theta=0$, these formulæ degenerate into those given in the last Article.

7. To express the complete values of

$$(a+b\sqrt{-1})^n+(a-b\sqrt{-1})^n.$$

$$(a+b\sqrt{-1})^n+(a-b\sqrt{-1})^n$$
$$=\rho\,\{\cos n\,(2r\pi+\theta)+\sqrt{-1}\sin n\,(2r\pi+\theta)\}$$
$$+\rho\,\{\cos n\,(2r\pi+\theta)-\sqrt{-1}\sin n\,(2r\pi+\theta)\}$$
$$=2\rho\cos n\,(2r\pi+\theta).$$

If n be a rational fraction, whose denominator is p, the values of r may be limited to the terms of the series

$$0,\ 1,\ 2\ldots\ldots p-1.$$

Thus, let $n=\frac{1}{3}$, then

$$(a+b\sqrt{-1})^{\frac{1}{3}}+(a-b\sqrt{-1})^{\frac{1}{3}}$$
$$=2\rho\cos\frac{\theta}{3},\ \text{or}\ 2\rho\cos\left(\frac{2\pi+\theta}{3}\right),\ \text{or}\ 2\rho\cos\left(\frac{4\pi+\theta}{3}\right),$$

where ρ is the arithmetic value of $(a^2+b^2)^{\frac{1}{6}}$.

The roots of the cubic equation

$$x^3-qx+r=0,$$

are expressed *generally* by the formula

$$x=\left\{\frac{-r}{2}+\sqrt{\left(\frac{r^2}{4}-\frac{q^3}{27}\right)}\right\}^{\frac{1}{3}}+\left\{\frac{-r}{2}-\sqrt{\left(\frac{r^2}{4}-\frac{q^3}{27}\right)}\right\}^{\frac{1}{3}}:$$

if we suppose $\frac{r^2}{4}$ to be less than $\frac{q^3}{27}$, and if we make

$$\cos\theta = \cos(2r\pi + \theta) = \frac{\frac{-r}{2}}{\sqrt{\frac{q^3}{27}}},$$

we shall find the three roots to be as follows:

$$x' = 2\rho\cos\frac{\theta}{3} = 2\sqrt{\frac{q}{3}}\cos\frac{\theta}{3}.$$

$$x'' = 2\rho\cos\left(\frac{2\pi+\theta}{3}\right) = 2\sqrt{\frac{q}{3}}\cos\left(\frac{2\pi+\theta}{3}\right).$$

$$x''' = 2\rho\cos\left(\frac{4\pi+\theta}{3}\right) = 2\sqrt{\frac{q}{3}}\cos\left(\frac{2\pi-\theta}{3}\right).$$

CHAP. IX.

ON THE EXPONENTIAL EXPRESSIONS FOR THE SINE AND COSINE OF AN ANGLE, AND ON SOME OF THE CONSEQUENCES DEDUCIBLE FROM THEM.

1. INASMUCH as

$$\cos^2\theta + \sin^2\theta = (\cos\theta + \sqrt{-1}\sin\theta)(\cos\theta - \sqrt{-1}\sin\theta) = 1,$$

$$\text{and } \cos(-\theta) = \cos\theta, \text{ and } \sin(-\theta) = -\sin\theta,$$

it follows, that if $\cos\theta + \sqrt{-1}.\sin\theta$ be expressed by ϵ^{θ}, then $\cos\theta - \sqrt{-1}\sin\theta$ will be expressed by $\epsilon^{-\theta}$.*

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* If there is any algebraical expression which is equivalent to

$$\cos\theta + \sqrt{-1}\sin\theta,$$

it must be an exponential expression, such as $\epsilon^{\phi(\theta)}$, where ϵ is a constant quantity (or a quantity not dependent upon θ), for the reason mentioned in the text: and the expression $\phi(\theta)$ can involve no fractional powers of θ, because

$$\cos\theta + \sqrt{-1}\sin\theta$$

has only *one* value for the *same* value of θ: and it can involve odd powers *only*

2. Assuming the representations of

$$\cos\theta + \sqrt{-1}\sin\theta \text{ and } \cos\theta - \sqrt{-1}\sin\theta,$$

given in the last Article, we get

$$\cos\theta = \frac{\epsilon^{\theta} + \epsilon^{-\theta}}{2} \text{ and } \sin\theta = \frac{\epsilon^{\theta} - \epsilon^{-\theta}}{2\sqrt{-1}}.$$

3. $\epsilon^{0} = \epsilon^{2r\pi} = 1.$

$$\epsilon^{\frac{\pi}{2}} = \epsilon^{(4r+1)\frac{\pi}{2}} = \sqrt{-1}.$$

$$\epsilon^{\pi} = \epsilon^{(2r+1)\pi} = -1.$$

$$\epsilon^{\frac{3\pi}{2}} = \epsilon^{(4r+3)\frac{\pi}{2}} = -\sqrt{-1}.$$

4. Since $\epsilon^{2r\pi} = 1$ and $\epsilon^{(2r+1)\pi} = -1$, we have

$$\epsilon^{2nr\pi} = (1)^{n} \text{ and } \epsilon^{(2r+1)n\pi} = (-1)^{n},$$

whatever the value of n may be.

5. $a^{\theta} = 1 + k\theta + \dfrac{k^{2}\theta^{2}}{1.2} + \dfrac{k^{3}\theta^{3}}{1.2.3} + \dfrac{k^{4}\theta^{4}}{1.2.3.4} + \text{\&c.}$ where

$$k = (a-1) - \frac{(a-1)^{2}}{2} + \frac{(a-1)^{3}}{3} - \frac{(a-1)^{4}}{4} + \text{\&c.}$$

both series being supposed to be continued *in infinitum.*

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of θ in every one of its terms, inasmuch as it changes its sign from + to −, when θ changes from + to −: consequently $\phi(\theta) = \theta\psi(\theta^{2})$: again, by Demoivre's formula,

$$\cos n\theta + \sqrt{-1}\sin n\theta = \left(\epsilon^{\theta\psi(\theta^{2})}\right)^{n} = \epsilon^{n\theta\psi(\theta)^{2}} = \epsilon^{n\theta\psi(n^{2}\theta^{2})};$$

and therefore $\psi(\theta)^{2} = \psi(n^{2}\theta^{2})$, whatever the value of n may be: consequently, $\psi(\theta^{2})$ is a quantity which does not change, for a change of the value of θ, and therefore it is a constant quantity: it follows from this reasoning, that

$$\cos\theta + \sqrt{-1}\sin\theta$$

may be correctly represented by ϵ^{θ}, where ϵ is to be determined by other considerations.

6. Let e be that value of a which makes $k=1$: the we have

$$e^{\theta} = 1 + \theta + \frac{\theta^2}{1.2} + \frac{\theta^3}{1.2.3} + \frac{\theta^4}{1.2.3.4} + \&c.$$

7. The value of e may be found from the series

$$1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \&c.;$$

and 2.7182818 is its correct value to 7 places of decimals.

8. Logarithms calculated to the base e, are called Napierian logarithms, and are almost exclusively used in analytical formulæ.

If the abbreviation log. presents itself in such formulæ, (when not reduced to logarithmic computation), Napieria logarithms are always understood, unless the contrary be expressed.

9. $k = \log a$.

10. $\frac{1}{k}$ is called the *modulus* of that system of logarithms which is calculated to the base a: in other words, it is the numerical quantity by which the Napierian logarithm of any number must be multiplied, in order to reduce it to the corresponding logarithm in the system whose base is a.

11. If $a=10$, then $\frac{1}{k} = \frac{1}{\log 10}$ = modulus of tabular logarithms, and its value is .43429448....

12. $\epsilon^{\theta} = 1 + (\log \epsilon)\,\theta + (\log \epsilon)^2 \frac{\theta^2}{1.2} + (\log \epsilon)^3 \frac{\theta^3}{1.2.3} + \&c.$

13. $\text{Cos}\,\theta = \frac{\epsilon^{\theta} + \epsilon^{-\theta}}{2} = 1 + (\log \epsilon)^2 \frac{\theta^2}{1.2} + (\log \epsilon)^4 \frac{\theta^4}{1.2.3.4} + \&c.$

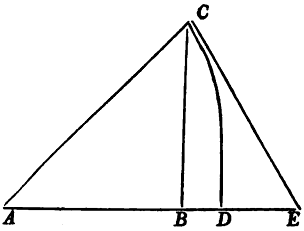
14. $\text{Sin}\,\theta = \frac{\epsilon^{\theta} - \epsilon^{-\theta}}{2\sqrt{-1}}$

$$= \frac{\log \epsilon}{\sqrt{-1}} \left\{ \theta + (\log \epsilon)^2 \frac{\theta^3}{1.2.3} + (\log \epsilon)^4 \frac{\theta^5}{1.2.3.4.5} + \&c. \right\}.$$

15. The preceding series for cos θ and sin θ, will satisfy the equation

$$\cos^2\theta + \sin^2\theta = 1,$$

whatever be the constant value of ϵ: it remains to consider in what manner the value of ϵ may be determined from the assumed measure of the angle θ.



16. If the measure of the angle BAC be θ, and if CD be an arc described from the centre C with the radius AC, then

$$\theta = \frac{\text{arc } CD}{AC}.$$

17. If CB be perpendicular to AB, and CE be perpendicular to AC, meeting AB produced in E, then we have

$$\sin\theta = \frac{BC}{AC} \text{ and } \tan\theta = \frac{BC}{AB} = \frac{CE}{AC}.$$

18. It may be shewn, from very simple geometrical considerations, that when CD is a small arc, BC is less and CE greater, than the intermediate arc CD, and therefore that sin θ is less than θ, but, that tan θ greater than θ.*

19. But

$$\sin\theta = \frac{\log\epsilon}{\sqrt{-1}}\,\theta + \frac{(\log\epsilon)^3\theta^3}{\sqrt{-1}.1.2.3} + \frac{(\log\epsilon)^5\theta^5}{\sqrt{-1}.1.2.3.4.5} + \&c.$$

$$\theta = \theta,$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\log\epsilon}{\sqrt{-1}}\,\theta - \frac{2\,(\log\epsilon)^3\theta^3}{\sqrt{-1}.1.2.3} + \&c.$$

But if there be three series arranged in the order of their magnitudes, proceeding according to integral powers of the same symbol (upon which also their values are dependent), the first terms of the first and third of which are the same;

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* *Algebra*, Art. 667.

then the first (and in this case the only) term of the second and intermediate series will be equal to the first term of the first or third series.*

Consequently, $\frac{\log \epsilon}{\sqrt{-1}} \theta = \theta$ and $\log \epsilon = \sqrt{-1}$;

and therefore $\epsilon = e^{\sqrt{-1}}$,† where e is the base of Napierian logarithms.

20. $$\operatorname{Cos} \theta = \frac{e^{\theta\sqrt{-1}} + e^{-\theta\sqrt{-1}}}{2}$$

$$= 1 - \frac{\theta^2}{1.2} + \frac{\theta^4}{1.2.3.4} - \frac{\theta^6}{1.2.3.4.5} + \&c.$$

$$\sin \theta = \frac{e^{\theta\sqrt{-1}} + e^{-\theta\sqrt{-1}}}{2\sqrt{-1}}$$

$$= \theta - \frac{\theta^3}{1.2.3} + \frac{\theta^5}{1.2.3.4.5} - \frac{\theta^7}{1.2.3.4.5.6.7} + \&c.$$

21. $$\epsilon^{2\theta\sqrt{-1}} = \frac{1 + \sqrt{-1} \,.\, \tan \theta}{1 - \sqrt{-1} \,.\, \tan \theta}.$$

22. $$\theta = \tan \theta - \frac{1}{3} \tan^3 \theta + \frac{1}{5} \tan^5 \theta - \&c. \textit{ in inf.}$$

If we suppose $\theta = \frac{\pi}{4} = 45^0$, we get

$$\frac{\pi}{6} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \&c.$$

If we suppose $\theta = \frac{\pi}{6} = 30^0$, we get

$$\frac{\pi}{6} = \frac{1}{\sqrt{3}} \left(1 - \frac{1}{3 \times 3} + \frac{1}{5 \times 3^2} - \frac{1}{7 \times 3^3} + \&c.\right)$$

This latter series would enable us to calculate the value of π to any required degree of accuracy.

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* *Algebra*, Art. 667. Note.

† *Algebra*, Art. 668.

CHAP. X.

UPON GENERAL SIGNS OF AFFECTION AND THEIR GEOMETRICAL INTERPRETATION.

1. THOSE symbolical expressions which do not affect the magnitude of the quantities into which they are incorporated, can be considered as *signs of affection* only or as equivalent to them, whether they admit of interpretation or not.

2. $+1$ and -1, employed as factors, merely affect the quantities into which they are incorporated, with the signs $+$ and $-$, in conformity with the general rule of those signs.

Quantities like $+a$ and $-b$, may be considered as the results respectively of the multiplication of $+1$ into a and of -1 into b.

3. Powers of $+1$ and of -1, such as $(+1)^n$ and $(-1)^n$, where n is any quantity whatsoever, may be considered as signs of affection.

If n be a whole number, these signs are necessarily equivalent to $+1$ or -1.

4. Inasmuch as

$$\cos 2nr\pi + \sqrt{-1} \, . \sin 2nr\pi = (1)^n, \text{ and}$$

$$\cos (2r+1) n\pi + \sqrt{-1} \, . \sin (2r+1) n\pi = (-1)^n,$$

where r is any whole number in the series $0, 1, 2 \ldots 3 \ldots$ &c., these expressions may be considered as signs of affection identical in signification with $(1)^n$ and $(-1)^n$ respectively.

5. The observation in the last Article applies to

$$e^{2nr\pi\sqrt{-1}} \text{ and } e^{(2r+1)n\pi\sqrt{-1}},$$

which are equivalent to

$$\cos 2nr\pi + \sqrt{-1} \, . \sin 2nr\pi$$

and $\cos (2r+1) n\pi + \sqrt{-1} \, . \sin (2r+1) n\pi$ respectively.

6. All signs of affection, which originate in the operations of Algebra, must either present themselves under, or be reducible to, the form $\alpha + \beta\sqrt{-1}$, where α and β may have any values whatever between 1 and -1, zero included; and where $\alpha^2 + \beta^2 = 1$*.

7. $a + b\sqrt{-1}$ and $a - b\sqrt{-1}$, where a and b are arithmetical or unaffected magnitudes, are equivalent to

$$(\cos\theta + \sqrt{-1}\sin\theta)\sqrt{a^2+b^2} \text{ and } (\cos\theta - \sqrt{-1}\sin\theta)\sqrt{a^2+b^2}$$

respectively, where

$$\cos\theta = \frac{a}{\sqrt{a^2+b^2}} = \cos\left(\cos^{-1}\frac{a}{\sqrt{a^2+b^2}}\right).$$

It thus appears, that such expressions as

$$a + b\sqrt{-1} \text{ and } a - b\sqrt{-1},$$

are equivalent to a *single* arithmetical or unaffected magnitude

$$\sqrt{(a^2+b^2)},$$

affected with the signs

$$\cos\theta + \sqrt{-1}\,.\sin\theta \text{ and } \cos\theta - \sqrt{-1}\,.\sin\theta \text{ respectively,}$$

$$\text{where } \theta = \cos^{-1}\frac{a}{\sqrt{(a^2+b^2)}}.$$

8. Two quantities, a and b, which are of the same kind when unaffected, one of which is affected with the sign $\alpha + \beta\sqrt{-1}$, are not equivalent to a single arithmetical or unaffected quantity, with a *determinate* sign of affection, unless $\alpha = 0$, and therefore $\beta = 1$.

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* The primitive signs of Algebra, which are + and −, or their equivalent factors +1 and −1, are sufficient to generalize all the primitive operations of Algebra, such as addition and subtraction, multiplication and division, when performed upon general symbols representing unaffected magnitudes; the only other derivative signs of affection are those which are introduced by involution or evolution, such as $(1)^n$ and $(-1)^n$ or their equivalents, where n may be a symbol equally general with any other symbol in Algebra.

This is a most important proposition, inasmuch as it precludes all attempts at the interpretation of compound expressions, such as

$$(\cos\theta + \sqrt{-1}\,\sin\theta)\,a + (\cos\theta' + \sqrt{-1}\,\sin\theta')\,b$$

$$= (\cos\theta + \sqrt{-1}\,\sin\theta)\,\{a + [\cos(\theta'-\theta) + \sqrt{-1}\,\sin(\theta'-\theta)]\,b\},$$

unless $\cos(\theta'-\theta) = 0$, and therefore $\sin(\theta'-\theta) = \pm 1$.

9. Principles of interpretation of derivative signs of affection.

(1) They must depend upon the specific nature of the magnitudes, to which they are applied.

(2) When a sign of affection is repeatedly applied to a symbol denoting a specific magnitude, producing in succession different symbolical results, in conformity with the laws of Algebra, their successive interpretations must be accordant with each other and with the interpretation of any resulting sign of affection, which has been previously determined.

Thus, if the repeated application of the sign of affection

$$\cos\theta + \sqrt{-1}\,\sin\theta,$$

should produce $+1$ or -1, as a necessary symbolical result, the interpretation of $\cos\theta + \sqrt{-1}\,.\sin\theta$, (when so applied,) must be such as to be accordant with that of $+1$ and -1, which may have been otherwise established.

(3) There is no *necessary* connection between a given symbolical result and its interpretation, though there may exist a necessary connection between the different interpretations of a series of symbolical results connected with each other.

If a denote a line, there does not exist necessarily an interpretation of

$$+a, \text{ or } -a, \text{ or } \sqrt{-1}\,.a, \text{ or } (\cos\theta + \sqrt{-1}\,.\sin\theta)\,a,$$

considered independently of each other.

The interpretation, however, of $(\cos\theta + \sqrt{-1}\,\sin\theta)\,a$, will necessarily determine that of $(\cos n\theta + \sqrt{-1}\,\sin n\theta)\,a$, if θ be a general symbol.

10. Examples of interpretations of signs of affection applied to symbols denoting geometrical magnitudes.

(1) Let a denote a straight line.

(α) If $+a$ denote a line drawn in one direction, $-a$ will denote a line drawn in the opposite direction, or making an angle of 180^0 with it.

This is the primitive interpretation of $+a$ and $-a$, when a denotes a straight line, with which all other interpretations of signs of affection, similarly applied, must be accordant.

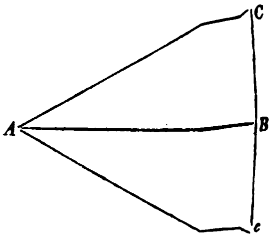
(β) If a denote a straight line, then

$$(\cos\theta + \sqrt{-1}\sin\theta)\, a$$

may denote a straight line of the same length, making an angle θ with the primitive line.

It immediately follows from hence, that $a\sqrt{-1}$ must denote a line of the length a, making an angle of 90^0 with the primitive line: and that $-a\sqrt{-1}$ must denote a line of the length a, making an angle of 270^0 with the primitive line.

(γ) If a and b denote two straight lines, then $a + b\sqrt{-1}$ will denote a line equal in length to $\sqrt{a^2+b^2}$, making an angle $\sin^{-1}\dfrac{b}{\sqrt{a^2+b^2}}$ with the primitive line: and $a - b\sqrt{-1}$ will denote a line making an angle equal to $-\sin^{-1}\dfrac{b}{\sqrt{a^2+b^2}}$ with the primitive line.



If $AB=a$ and $BC=b\sqrt{-1}$,

$$\text{then } AC = a + b\sqrt{-1}$$
$$= (\cos\theta + \sqrt{-1}\sin\theta)\sqrt{a^2+b^2},$$
$$\text{where } \theta = \sin^{-1}\frac{b}{\sqrt{a^2+b^2}} = BAC:$$

and if $AB=a$ and $Bc = -b\sqrt{-1}$,

$$\text{then } Ac = a - b\sqrt{-1} = (\cos\theta - \sqrt{-1}\sin\theta)\sqrt{a^2+b^2},$$

$$\text{where } \theta = \sin^{-1}\frac{b}{\sqrt{a^2+b^2}}.$$

11. The algebraical *sum* of two lines drawn in given directions, is the diagonal of the parallelogram which they would form if drawn from the same point parallel to those directions, and which they would include: their algebraical *difference* is the other diagonal which they would not include.

12. The algebraical *sum* of the sides of a *closed* figure or a series of lines, which, if placed successively in contact, and parallel to the given lines, would form a *closed* figure, is *zero*.

The equation which expresses this condition is called the *general equation of figure*.

13. The algebraical sum of the exterior angles of a *closed* figure must be either *four* right angles or a multiple of four right angles.

This condition of a closed figure forms the *general equation of angles*.

The figures which are usually considered in Geometry are those in which the sum of the exterior angles is *four* right angles, and from which all negative exterior angles are excluded.

14. The preceding propositions are sufficient to bring Geometry, as far as relates to rectilinear figures in the same plane and their properties, entirely under the dominion of Algebra.

CHAP. XI.

ON THE APPLICATION OF ALGEBRA TO GEOMETRY, BY CONSIDERING THE POSITION OF INDEPENDENT AND DEPENDENT POINTS IN SPACE.

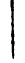
Sect. 1.

Position of a Point and its Co-ordinates.

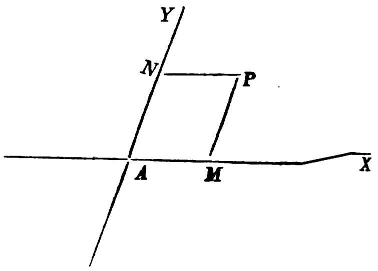
1. A Point in space is said to be given, when it is subject to conditions, which belong to that point and to no other.

2. A point in space is given in position, if its perpendicular distances from two given lines or *axes*, in the same plane with it, are given.

3. If AX and AY be two lines or *axes*, which are at right angles to each other and in the same plane with the point P: and if PM and PN be perpendicular to AX and AY respectively: then PM and PN will *determine* the position of the point P.



4. If AX and AY be two lines or axes *not* at right angles to each other and in the same plane with the point P: and if PM and PN be drawn parallel to AY and AX respectively: then PM and PN will *determine* the position of the point P.



Axes of the first kind are called *rectangular*, and of the second kind *oblique:* rectangular axes are those which are always understood to be used, unless the contrary be expressed.

5. Since $PN = AM$, in both the preceding figures, the position of the point P is determined by AM and MP.

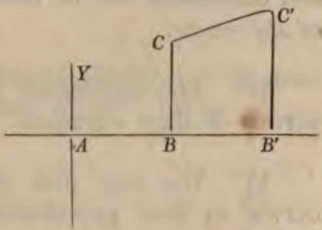
6. It is usual to call AM, which is reckoned upon or cut *off* from, the axis AX, the *abscissa*, and the line PM, which is parallel to the other axis AY, the *ordinate:* but when they are mentioned together, and are not required to be distinguished from each other, they are called the *co-ordinates* of the point P. The point A, where the axes intersect each other, is called the *origin* of the co-ordinates.

7. It is usual when the position of the point P is given, but *unknown*, or when the point P is not *fixed*, but considered in different and *variable* positions, to denote the abscissa AM by x, and the ordinate MP by y: and the axis AX is usually called the axis of the *abscissæ* or of x, and the other axis AY is called the axis of the *ordinates* or of y.

If the position of the point P be given, its co-ordinates are usually denoted by the earlier letters of the alphabet, or by x and y accented in such a manner as to distinguish them from x and y when denoting unknown or variable co-ordinates.

8. Enumeration of the different positions of the point P, in the four quadrants formed by the intersections of the two axes, corresponding to all the different combinations of $+x$, $-x$, and $+y$, $-y$.

9. If there be two given points, their distance from each other, and the angle of inclination to the axis of x of the line which joins them, may be determined from their respective co-ordinates.



If α and β, α' and β' be the co-ordinates of two given points C and C', then their distance $(\delta) = \sqrt{\{(\alpha - \alpha')^2 + (\beta - \beta')^2\}}$, and the tangent of its inclination (θ) to the axis of $x = \frac{\beta' - \beta}{\alpha' - \alpha}$.

10. The position of a point P is said to be *arbitrary* or *independent*, when its co-ordinates are *arbitrary* or *independent*, whether they be *known*, *unknown*, or *variable*.

11. The position of a point is said to be *dependent*, when its co-ordinates are *dependent* upon each other; or when the value of one of them necessarily determines that of the other.

12. In Geometry, the position of a point P is said to be *dependent*, when its different positions are subject to the condition of being always found in a given line or curve, or required to satisfy some prescribed law of movement.

13. In Algebra, the position of a point P is said to be *dependent*, when the symbols which represent its co-ordinates for all different positions, must satisfy an equation in which they are involved, with or without other known or determinate quantities.

14. In such an equation involving x and y, for every value of x there will be at least *one** value of y, and conversely: and if all such values of x and y be considered, the point to which they correspond must describe a curve.

15. The equation which connects the values of x and y, and determines one of them from the other, is the *equation of the curve.*

16. The algebraical properties of curves are the properties of their equations.

17. We may pass from the algebraical properties curves to their geometrical properties, and conversely, means of interpretations made conformably with the laws Algebra.

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* If values which are called *impossible* or *imaginary*, as well as possible valu be included.

SECT. 2.

Equation to a straight line and its immediate consequences.

18. Let the *locus* of the point P be a straight line as defined in Geometry; and let it be required to determine its equation.

A straight line is determined geometrically; *first*, if it pass through a given point and make a given angle with a given straight line or *axis*: *secondly*, if it pass through two given points.

19. For the first case.

Let α and β be the co-ordinates of the given point, and x and y the co-ordinates of any other point whatsoever in the line whose equation is required; and let θ be the angle of its inclination to the axis of x: then

$$\tan\theta = \frac{y-\beta}{x-\alpha}, \text{ (Art. 9.)}$$

$$\text{or } y-\beta = \tan\theta\,(x-\alpha),$$

$$\text{or } y = ax + b,$$

$$\text{if } a = \tan\theta \text{ and } b = \beta - \alpha\tan\theta.$$

20. For the second case.

Let α and β, α' and β' be the co-ordinates of the two given points, through which the line, whose equation is required, passes; then, since

$$\tan\theta = \frac{\beta'-\beta}{\alpha'-\alpha}, \text{ (Art. 9.)}$$

we have, by the last Article,

$$y-\beta = \frac{\beta'-\beta}{\alpha'-\alpha}(x-\alpha).$$

21. The most general equation of the first order involving two variable quantities x and y, is

$$Ay + Bx + C = 0,^{*}$$

which is in all cases the equation to a straight line.

It is reducible to the form

$$y - ax - b = 0, \text{ or } y = ax + b, \text{ (Art. 19.)}$$

by making $a = \frac{-B}{A}$, and $b = \frac{-C}{A}$.

22. A straight line may, in all cases, be determined from its equation

$$y = ax + b,$$

when the quantities a and b are given.

Examples of the determination of lines from their equations.

If $x = 0$, what is the value of y?
If $y = 0$, what is the value of x?

The answers to these two questions will give two points, one in the axis of y and the other in the axis of x, through which the line required must pass: if, however, the equation be $y = ax$, the line passes through the origin of the co-ordinates, and only one point in the line is thus determined: a second point in the line may be found by the answer to the following question: If $x = 1$, (or any other given line), what is the value of y?

23. Geometrical proposition. If two straight lines be given, which are in the same plane, their inclination to each other and their common point of intersection are given.

24. Algebraical proposition. Given the equations of two straight lines, to find the angle between them and the co-ordinates of their point of intersection.

Let

$$y = ax + b,$$
$$y' = a'x' + b',$$

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[*] If A, B and C be considered as arbitrary constant quantities, one of them superfluous.

be the equations of the two given lines, in which a and b, a' and b' are therefore given: let ϕ be the angle of their inclination to each other, and let α, β be the co-ordinates of their point of intersection: then

$$\tan\phi = \frac{a'-a}{1+aa'}, \qquad \sin\phi = \frac{a'-a}{\sqrt{1+a^2}\sqrt{1+a'^2}},$$

$$\cos\phi = \frac{1+aa'}{\sqrt{1+a^2}\sqrt{1+a'^2}},$$

$$\alpha = -\frac{b'-b}{a'-a}, \qquad \beta = \frac{a'b-ab'}{a'-a}.$$

25. If $a = a'$, or if the coefficients of x in the two equations be *equal*, then $\phi = 0$, and α and β are both of them infinite; or in other words, the lines are parallel, and therefore never meet.

26. If $a' = -\frac{1}{a}$, or if the coefficient of x in one equation be the reciprocal of the coefficient of x in the other equation, with its sign changed from + to −, or conversely; then $\phi = 90^\circ$, and the two lines are at right angles to each other: under the same circumstances also,

$$\alpha = \frac{a(b'-b)}{1+a^2} \quad \text{and} \quad \beta = \frac{b+a^2b'}{1+a^2}.$$

27. Geometrical proposition. To draw a line which shall pass through a given point, and which shall make a given angle with a given straight line.

28. Algebraical proposition. To find the equation to a straight line, which shall pass through a point whose co-ordinates are given, and which shall make a given angle with a line whose equation is given.

Let α and β be the co-ordinates of the given point, and $\phi = \tan^{-1} t$ the angle which the line whose equation is required shall make with the line whose equation $y = ax + b$, is given: then

$$y' - \beta = \frac{a+t}{1-at}(x'-\alpha). \text{ (Art. 19.)}$$

If t be infinite, or $\phi = 90^0$, we get

$$y' - \beta = -\frac{1}{a}(x' - \alpha),$$

a conclusion which agrees with that given in Art. 26.

29. Geometrical proposition. If a straight line be drawn from a *given* point, and making a *given* angle with a *given* straight line, its length and the point of its intersection with the given line are given.

30. Algebraical proposition. Given the equation of a straight line and the co-ordinates of a point without it: to find the length of a line drawn from the point whose co-ordinates are given to the line whose equation is given and making a given angle with it, and likewise the co-ordinates of the point of intersection.

Let α and β be the co-ordinates of the given point, and $y = ax + b$ the equation of the given line: let a' be the tangent of the angle which the line whose length is required makes with the axis of x: and let α' and β' be the co-ordinates of the point of intersection, and d the length of the line: then

$$\alpha' = \frac{\beta - b - a'\alpha}{a - a'},$$

$$\beta' = \frac{a\beta - a'(a\alpha + b)}{a - a'},$$

$$d = -\frac{(\beta - b - a\alpha)\sqrt{1 + a'^2}}{a' - a}\,*$$

$$= \frac{(\beta - b)\cos\theta - \alpha\sin\theta}{\sin(\theta' - \theta)},$$

if $a = \tan\theta$ and $a' = \tan\theta'$.

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* We have assumed the negative sign in passing from the square to the square root, in order to obtain the value of d.

31. If $\theta' = 90 + \theta$, or if the line drawn from the given point be perpendicular to the given line, the formulæ in the last Article become, since $a' = -\frac{1}{a}$,

$$\alpha' = \frac{a(\beta - b) + \alpha}{1 + a^2},$$

$$\beta' = \frac{a^2\beta + a\alpha + b}{1 + a^2},$$

$$d = \frac{\beta - b - a\alpha}{\sqrt{(1 + a^2)}}$$

$$= (\beta - b)\cos\theta - \alpha\sin\theta.$$

32. If the line, upon which the perpendicular is let fall, passes through the origin of the co-ordinates, the formulæ in the last Article become

$$\alpha' = \frac{a\beta + \alpha}{1 + a^2},$$

$$\beta' = \frac{a^2\beta + a\alpha}{1 + a^2},$$

$$d = \frac{\beta - a\alpha}{\sqrt{(1 + a^2)}}$$

$$= \beta\cos\theta - \alpha\sin\theta.$$

SECT. 3.

Transformation of Co-ordinates.

33. When the equation which expresses the relation between the co-ordinates of a *dependent* point is determined with respect to assumed axes, it is frequently required to *transform* this equation in such a manner, that it may express the relation between the co-ordinates of the *same* point subject to the *same* conditions, when referred to *new* axes whose positions are given with respect to the former.

34. Let x' and y' be the new co-ordinates, x and y the old; and let θ be the inclination of the axis of x' to that of x, and therefore $-\theta$ the inclination of the axis of x to that of x': consequently, the equations of the old axes with respect to the new, are

$$y = ax + b = -\tan\theta \times x + b,$$

$$y = -\frac{1}{a}x + b' = \frac{x}{\tan\theta} + b',$$

respectively: and since x and y are the lengths of the perpendiculars upon these lines respectively, we get by the formulæ in Art. 31.

$$y = \frac{y' - ax' - b}{\sqrt{1 + a^2}}$$

$$= x' \sin\theta + y' \cos\theta - b\cos\theta,$$

$$x = \frac{y' + \frac{x'}{a} - b'}{\sqrt{\left(1 + \frac{1}{a^2}\right)}}$$

$$= x' \cos\theta - y' \sin\theta - b' \sin\theta.$$

The substitution of these values of x and y, in the equ tion between them, will effect the transformation required.

35. If the new axes have the same origin with the ol then $b = 0$ and $b' = 0$, and the expressions in the last Articl become

$$x = x' \cos\theta - y' \sin\theta,$$

$$y = x' \sin\theta + y' \cos\theta.$$

36. If c and c' be the co-ordinates of the new *origi of the co-ordinates* with respect to the old, then (inasmu as at that point $x' = 0$ and $y' = 0$,) we get from the formul in Art. 34.

$$c = -b' \sin\theta, \qquad c' = -b\cos\theta.$$

37. If it be required to express the new co-ordinates in terms of the old, we shall find

$$x' = (x - c)\cos\theta + (y - c')\sin\theta,$$
$$y' = (x - c)\sin\theta - (y - c')\cos\theta.$$

38. The expressions in the last Article become, if $c = 0$ and $c' = 0$,

$$x' = x\cos\theta + y\sin\theta,$$
$$y' = x\sin\theta - y\cos\theta.^{*}$$

39. In effecting transformations of an equation, when *both* the origin and inclination of the axes are changed, it is generally most convenient to transform the equation, *first*, for a change of the origin *only*, without a change of the inclination of the axes, and *secondly*, for a change of the inclination of the axes *only*, without a change of the origin of the co-ordinates.

For the first transformation, we make

$$x = x' + c,$$
$$y = y' + c',$$

where c and c' are the co-ordinates of the new origin.

For the second transformation, we make

$$x' = x''\cos\theta - y''\sin\theta,$$
$$y' = x''\sin\theta + y''\cos\theta,$$

where θ is the inclination of the new axis of x to the old.

---

[*] The pair of equations

$$x = ax' + by',$$
$$y = bx' - ay',$$

are convertible into the pair of equations

$$x' = ax + by,$$
$$y' = bx - ay,$$

if $a^2 + b^2 = 1$.

In the final result, the accents may, generally speaking, be suppressed, unless different forms of the equation of the same curve are required to be considered with reference to each other.

40. If it be required to transform the equation of a curve from *rectangular* to *oblique* axes, having the same origin of the co-ordinates, and which make angles θ and θ' with the axis of x and therefore the angle $\theta' - \theta$ with each other, we must make

$$\left.\begin{array}{l} x = x' \cos\theta + y' \cos\theta' \\ y = x' \sin\theta + y' \sin\theta' \end{array}\right\}.$$

For it appears from Art. 30, making $b = 0$, and replacing α and β by x and y, that

$$y' = \frac{y\cos\theta - x\sin\theta}{\sin(\theta' - \theta)},$$

$$x' = \frac{x\sin\theta' - y\cos\theta'}{\sin(\theta' - \theta)},$$

which are convertible into the equations

$$x = x' \cos\theta + y' \cos\theta',$$

$$y = x' \sin\theta + y' \sin\theta'.$$

This is a most important transformation, and includes the one given in Art. 34.

41. Let it be required to transform the equation

$$y = ax + b,$$

of a straight line (where $a = \tan\phi$), from rectangular to oblique axes, having the same origin, and making the angles θ and θ' with the axis of x.

The new equation becomes

$$y' \sin(\theta' - \phi) + x' \sin(\theta - \phi) - b\cos\phi = 0.$$

42. The propositions considered in the preceding articles, shew in what manner all problems relating to the positions of straight lines in space, and their inclinations to each other and to given axes, as well as to the positions of their points of intersection, may admit of an algebraical solution.

The direct representation of lines in space, considered in Chapter X, refers them to *one axis only*, and therefore fails in affording the solution of many geometrical problems, where the absolute as well as the relative positions of lines, with respect to each other, are required to be considered.

SECT. 4.

Equation of a Circle.

43. Geometrical definition of a circle. A circle is a curve, in the same plane, described by a point which is always at the same distance from the same given point.

44. Algebraical problem. To find the equation which expresses the relation of *dependence* of the co-ordinates of the describing point of a circle.

Let α and β be the co-ordinates of the centre or fixed point, x and y the co-ordinates of the describing point, and γ the invariable distance: then (Art. 9.)

$$(x-\alpha)^2 + (y-\beta)^2 = \gamma^2 \ldots\ldots (\alpha).$$

45. If the origin of the co-ordinates be changed to the centre of the circle, the *transformed* equation will become

$$x^2 + y^2 = \gamma^2 \ldots\ldots\ldots\ldots\ldots\ldots (\beta).$$

46. If $\alpha = \gamma$ and $\beta = 0$, then the equation becomes

$$2\gamma x - x^2 = y^2 \ldots\ldots\ldots\ldots\ldots (\gamma).$$

In this case, the origin of the co-ordinates is at the extremity of a diameter, which coincides with the axis of x.

47. The equation

$$x^2 + y^2 + ax + by + c = 0,$$

is the equation of a circle, the co-ordinates of whose centre are $\frac{-a}{2}$ and $\frac{-b}{2}$, and whose radius $= \frac{1}{2}\sqrt{(a^2 + b^2 - 4c)}$.

If $a^2 + b^2 = 4c$, the circle or the curve denoted by the equation, as far as it is included in the plane of x, y, degenerates into a point: if $a^2 + b^2$ be less than $4c$, no part of the curve, expressed by the equation, is found in the plane of x, y.

The equation of the circle can comprehend no term involving the product xy, unless the axes to which it is referred are oblique.

In the discussion which follows, of curves determined from their equations, we shall confine our attention to those parts of them which are situated in the plane of the axes of x and y: in other words, we shall consider those values of x and y which are affected with the sign $\sqrt{-1}$, or with any sign of affection different from $+$ or $-$, as *imaginary*, and as belonging to points which have no connection with the curves represented by the equations which may come under our consideration.

SECT. 5.

Discussion of the general equation of Curves of the second order.

48. Curves of the second order are those, whose equations, under their most simple *rational* form, involve squares or simple products of their co-ordinates, and no powers or products of them which are of higher dimensions.

49. The *most general* form of an equation of the *secon* order, involving x and y, is

$$ay^2 + bxy + cx^2 + dy + ex + f = 0 \ldots\ldots (a).$$

In this equation there is one superfluous *arbitrary coefficient* or *constant*, which might be suppressed; there are, however, some advantages which result from retaining it, when discussing the general properties of the curves which the equation may represent.

Our first enquiry will be, to determine to what extent the form of this equation may be simplified by the transformation of co-ordinates; our second enquiry will refer to the distribution of the curves which it may represent into their proper classes.

50. The terms dy and ex, or those involving the simple powers of x and y only, may be exterminated, *by changing the origin of the co-ordinates only, the new axes remaining parallel to the old*, in all cases, unless $b^2 - 4ac = 0$.

The transformed equation is

$$ay'^2 + bx'y' + cx'^2 + f' = 0 \ldots\ldots\ (b),$$

where the three first coefficients of (a) remain unaltered, and where

$$f' = a\beta^2 + b\alpha\beta + c\alpha^2 + d\beta + e\alpha + f,$$

$$\text{making } \alpha = \frac{2ae - bd}{b^2 - 4ac} \text{ and } \beta = \frac{2cd - be}{b^2 - 4ac},$$

where α and β are the co-ordinates of the *new* origin.

51. The exception above mentioned, where $b^2 - 4ac = 0$, and where the terms involving the simple powers of x and y cannot be exterminated *simultaneously*, indicates the existence of a class of curves, *without centres*, (Art. 56.), included in equation (a), to which the general name of *parabola* is commonly given.

52. The equation

$$ay^2 + bxy + cx^2 + f' = 0 \ldots\ldots\ldots\ldots (b),$$

(suppressing the accents of x and y), may be transformed *in all cases*, in such a manner, that the term involving xy may be exterminated, by ***changing the direction of the rectangular axes*** (*without altering the origin*) ***through*** an angle θ, such that $\tan 2\theta = \frac{-b}{a-c}$.

The transformed equation becomes (suppressing the accents of x and y),

$$a'y^2 + c'x^2 + f' = 0 \ldots\ldots\ldots (c),$$

or

$$\tfrac{1}{2}\{a+c+\sqrt{[(a-c)^2+b^2]}\}y^2+\tfrac{1}{2}\{a+c-\sqrt{[(a-c)^2+b^2]}\}x^2+f'=0 \; (d).$$

53. If $4ac$ be greater than b^2, the coefficients a' and c' must have the same algebraical sign, whether + or − : if $4ac$ be less than b^2, the coefficients a' and c' must have different algebraical signs, whether + or − : if $4ac = b^2$, we have $c' = 0$: but this is the case excepted above (Art. 50.), where the coefficients of the simple powers of x and y have not disappeared simultaneously; if $a = c$ and also $b = 0$, the value of θ is indeterminate, and the general equation becomes that of a circle.

54. If the two transformations in Arts. 50. and 52. are made *simultaneously*, the values of θ, α and β which result, will be the same as those determined: if the two transformations are effected in a contrary order, the value of θ will be the same as before, but the co-ordinates (α', β') of the new origin being referred to the new and not to the old axes, will be

$$\alpha' = -\frac{(d \sin\theta + e\cos\theta)}{a + c - \sqrt{\{(a-c)^2 + b^2\}}},$$

$$\beta' = -\frac{(d \cos\theta - e\sin\theta)}{a + c + \sqrt{\{(a-c)^2 + b^2\}}}.$$

55. To find the points of intersection of the line whose equation is $y = mx$, with the curve or curves expressed by the equation

$$a'y^2 + c'x^2 + f' = 0 \ldots\ldots\ldots (e).$$

The co-ordinates of the points of intersection are

$$\pm\sqrt{\left(\frac{-f'}{m^2a'+c'}\right)} \quad \text{and} \quad \pm m\sqrt{\left(\frac{-f'}{m^2a'+c'}\right)},$$

which are both + or both − ; the points are therefore equidistant from the origin of the co-ordinates.

56. The origin of the co-ordinates, in the curve represented by the equation

$$a'y^2 + c'x^2 + f' = 0,$$

is called a *centre,* inasmuch as all lines passing through it and meeting the curve both ways, are bisected in that point.

57. Classification of the curves represented by the equation

$$a'y^2 + c'x^2 + f' = 0,$$

which results from an examination of the values of m in Art. 55.

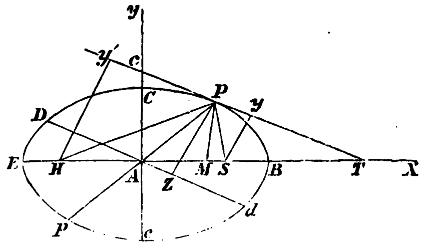
58. Let a', c', f' have the same algebraical sign, whether + or −.

The values of m involve $\sqrt{-1}$, if those of x do not involve $\sqrt{-1}$; and there is no part, therefore, of the corresponding curve which is found in the plane of x, y.

59. Let a' and c' have the *same* sign, whether + or −, which is different from that of f'.

The values of m are unlimited, whether + or −.

In this case, the curve in the plane of x, y, surrounds the centre, at a finite distance, in every direction; forming a species of *oval,* which is called an *ellipse.*



This is the second great class of curves included under the general equation (a). Art. 49.

60. If a' and c' be equal, the *ellipse* degenerates into a *circle*. Art. 46.

61. If AB or the greatest value of x (when $y=0$) be denoted by a, and if AC or the greatest value of y (when $x=0$) be denoted by b, then

$$a^2=\frac{-f'}{c'} \text{ and } b^2=\frac{-f'}{a'},$$

and the equation (c), Art. 52. becomes

$$\frac{y^2}{b^2}+\frac{x^2}{a^2}=1 \ldots\ldots\ldots\ldots\ldots (e),$$

$$\text{or } a^2y^2+b^2x^2=a^2b^2 \ldots\ldots\ldots\ldots (f).$$

The values of a and b in this equation must not be confounded with the values of a and b in the general equation (a).

62. a and b are called the *semiaxes* of the ellipse, and $2a$ and $2b$, or Be and Cc, are called the *axes:* the greater is called the axis *major*, and the less the axis *minor*.

63. If a' and c' have different signs, the limits of m are

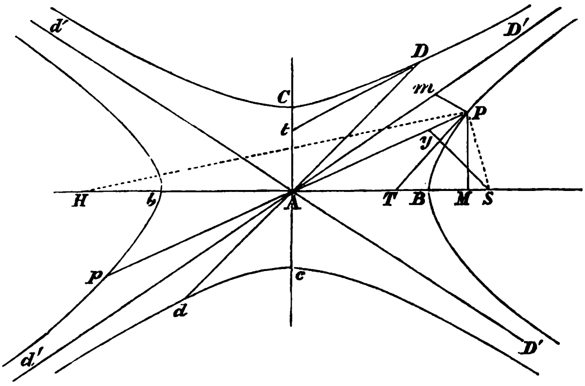
$$\pm\sqrt{\left(\frac{-c'}{a'}\right)} \text{ or } \frac{\pm\sqrt{(b^2-4ac)}}{a+c+\sqrt{\{(a-c)^2+b^2\}}}.$$

If a' and f' have the *same* sign, those limits are *superior* limits, which m cannot exceed: if a' and f' have *different* signs, these limits are *inferior* limits, than which m must always be greater: for greater values of m in one case or for less values of m in the other, the line whose equation is $y=mx$, will not meet the curve.

Let the lines DAd' and $D'Ad$, whose equations are

$$y=x\sqrt{\left(\frac{-c'}{a'}\right)} \text{ and } y=-x\sqrt{\left(\frac{-c'}{a'}\right)}$$

respectively, be drawn: let $AB = \sqrt{\left(\frac{-f'}{c'}\right)}$ in the first case, and $AC = \sqrt{\left(\frac{-f'}{a'}\right)}$ in the second; make $Ab = AB$ and $Ac = AC$: the curves in the first case are *double*, having their *vertices* at B and b, and included between the angles DAD' and dAd': the curves in the second case are also *double*, having their vertices at C and c, and included within the angles DAd and $D'Ad'$: their branches are infinite, the values of x and y being unlimited, both $+$ and $-$.



These curves are called by the common name of *hyperbola:* the first pair of curves (whose vertices are at B and b) are said to be *conjugate* to the second pair (whose vertices are at C and c), and conversely. They form the *third* great class of curves, which are included under the general equation (a).

64. If $AB = a$ and $AC = b$, the equation (c) becomes

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = -1, \text{ or } a^2y^2 - b^2x^2 = -a^2b^2 \ldots\ldots (g),$$

in the first case, and

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1, \text{ or } a^2y^2 - b^2x^2 = a^2b^2 \ldots\ldots\ldots\ldots (h),$$

in the second.

65. a and b are called the ***semiaxes***, $2a$ and $2b$ or Bb and Cc the ***axes;*** the greater being called the ***axis major*** and the less the ***axis minor*** of the hyperbola.

66. The limiting lines DAd' and $D'Ad$ are called *asymptotes* of the hyperbolas: the curves approach infinitely near to these asymptotes, but never touch them.

67. An *asymptote* generally, is a straight line whose equation coincides with the equation of the curve at an *infinite* distance: and no curve can admit of an asymptote whose equation does not *degenerate* into that of a straight line under such circumstances.

An ellipse can have no asymptote: a *parabola* (whose equation will be afterwards considered) can have no asymptote.

68. A *diameter* of a curve is a line which bisects all parallel chords (or lines joining two points of the curve).

69. The major and minor axes or those axes produced, of the hyperbola and of the ellipse, are diameters.

70. The major and minor axes are the only rectangular diameters (that is, which cut the chords at right angles,) which those curves admit of.

71. Let it be required to determine whether there exist any *oblique* diameters of the ellipse and hyperbola.

For this purpose, let it be required to transform the equation

$$Ay^2 + Bx^2 = 1 \ldots\ldots\ldots\ldots\ldots (i)$$

from rectangular to oblique axes, making angles θ and θ' with the axis of x: if the transformed equation can be reduced to the form

$$A'y'^2 + B'x'^2 = 1 \ldots\ldots\ldots\ldots\ldots (k,)$$

wanting the term which involves $x'y'$, then the new *axes* are *diameters*, cutting the chords at the angles of inclination of the axes to each other.

72. If $\tan\theta \tan\theta' = \frac{-B}{A} = \frac{-b^2}{a^2}$, for the ellipse, and $\frac{b^2}{a^2}$ for the hyperbola, then the transformed equation will assume the form

$$A'x'^2 + B'y'^2 = 1;$$

$$\text{where } A' = A\sin^2\theta' + B\cos^2\theta',$$

$$B' = A\sin^2\theta + B\cos^2\theta.$$

73. These new axes are therefore *diameters*, and when considered with reference to each other, are called *conjugate* diameters: there is an infinite number of conjugate diameters.

74. Since $\tan\theta = \frac{y}{x}$, we have $\tan\theta' = \frac{-b^2}{a^2} \cdot \frac{x}{y}$ in the ellipse, and $\frac{b^2}{a^2} \cdot \frac{x}{y}$ in the hyperbola, and therefore

$$\tan(\theta' - \theta) = \frac{-a^2b^2}{(a^2 - b^2)\,xy} \text{ in the ellipse,}$$

$$= \frac{a^2b^2}{(a^2 + b^2)\,xy} \text{ in the hyperbola.}$$

The angle $\theta' - \theta$, is the angle between the conjugate diameters, and it is also the angle of *ordination* of the chords: it is given, therefore, for every given point of the curve.

75. If d and d' represent the conjugate diameters, the equation of the ellipse with respect to them is

$$d^2y'^2 + d^{2\prime}x^2 = d^{2\prime}d^2 \ldots\ldots\ldots\ldots\ldots (l),$$

and the equations of the hyperbola and its conjugate are

$$d^2y'^2 - d'^2x^2 = \pm\, d^2d'^2 \ldots\ldots\ldots\ldots (m).$$

76. If we transform the equations (l and m) from oblique to rectangular axes, by making (Art. 40.)

$$x' = \frac{x \sin \theta' - y \cos \theta'}{\sin (\theta' - \theta)} \text{ and } y' = \frac{y \cos \theta - x \sin \theta}{\sin (\theta' - \theta)},$$

we shall reproduce the equation (*f*): and by equating the corresponding coefficients of the identical equations which result, we shall find, for the ellipse

$$\left.\begin{array}{l} d^2 \cos^2 \theta + d'^2 \cos^2 \theta' = a^2 \\ d^2 \sin^2 \theta + d'^2 \sin^2 \theta' = b^2 \end{array}\right\} \ldots\ldots\ldots\ldots (n),$$

$$d^2 d'^2 \sin^2 (\theta' - \theta) = a^2 b^2 \ldots\ldots\ldots\ldots (o),$$

$$d^2 \sin 2\theta + d'^2 \sin 2\theta' = 0 \ldots\ldots\ldots\ldots\ldots (p);$$

and for the hyperbola

$$\left.\begin{array}{l} d^2 \cos^2 \theta - d'^2 \cos^2 \theta' = a^2 \\ - d^2 \sin^2 \theta + d' \sin^2 \theta' = b^2 \end{array}\right\} \ldots\ldots\ldots\ldots (n'),$$

$$d^2 d'^2 \sin^2 (\theta' - \theta) = a^2 b^2 \ldots\ldots\ldots\ldots (o'),$$

$$d^2 \sin 2\theta - d'^2 \sin^2 2\theta' = 0 \ldots\ldots\ldots\ldots\ldots (p').$$

In speaking of the properties of the conjugate diameters of an hyperbola, we always understand the whole system of curves formed by the opposite hyperbolas and their conjugates.

77. In the ellipse,

$$a^2 + b^2 = d^2 + d'^2 \ldots\ldots\ldots\ldots\ldots\ldots (r),$$

or the sum of the squares described upon the principal axes is equal to the sum of the squares described upon *any* two conjugate diameters.

In the hyperbola,

$$a^2 - b^2 = d^2 - d'^2 \ldots\ldots\ldots\ldots\ldots\ldots (r'),$$

or the difference of the squares described upon the princip axes is equal to the difference of the squares described upo any two conjugate diameters.

78. In both the ellipse and hyperbola,

$$dd' \sin (\theta' - \theta) = ab;$$

or the rectangle described upon the principal axes is equa to the parallelogram which is formed by drawing lines throug he extremities of each of two conjugate diameters parallel in ach case to the corresponding conjugate diameter.

Inasmuch as it will be found, that lines so drawn are tanents to the ellipse or hyperbola, it follows that all parallelgrams circumscribing an ellipse, and all parallelograms touchig a system of conjugate hyperbolas, will be equal to each ther.

79. Definition. A tangent to a curve, at a given point, a straight line which meets the curve at that point, and eing produced, does not cut the curve at any point *immeiately* preceding or following.

The phrase *immediately preceding or following* may be replaced y the following: *at any distance whatever from the point of contact, wever small it may be.*

80. The tangent to any point of an ellipse or hyperbola parallel to the *conjugate* diameter corresponding to that oint.

This conclusion follows from very simple geometrical considertions: for if the tangent cuts the curve in any point different om the point of contact, it forms a chord which is *not* one of he system of chords to which the tangent, as drawn, is necessarily arallel.

81. The inclination (θ') of the tangent to the ellipse hyperbola to the axes is the same as that of the conjugate ameter to which it is parallel; and therefore $\tan\theta' = -\frac{b^2x}{a^2y}$ the ellipse, and $\frac{b^2x}{a^2y}$ for the hyperbola, (Art. 74.)

82. The equation of the tangent is, for the ellipse,

$$a^2yy' + b^2xx' = a^2b^2,$$

d for the hyperbola,

$$a^2yy' - b^2xx' = \pm a^2b^2;$$

where x' and y' are the co-ordinates of any point in the tangent, x and y being the co-ordinates of the point of contact.

83. The distances from the centre, AT and At whe the tangent cuts the axes of x and y respectively, are and $\frac{b^2}{y}$ respectively.

The distance AT upon the axis of x is independent the axis minor, and the distance At upon the axis of y independent of the axis major.

If, therefore, a circle be described upon the major axis of ellipse, and the ordinate of the ellipse be produced to meet the tangent to this point of the circle will meet the axis of x the major axis produced, at the same point with the tangent fr the corresponding point of the ellipse.

84. A normal to a curve at any point is a line whic is perpendicular to the tangent at that point.

85. The equation of the normal is, for the ellipse,

$$\frac{a^2x'}{x} - \frac{b^2y'}{y} = a^2 - b^2,$$

and for the hyperbola,

$$\frac{a^2x'}{x} + \frac{b^2y'}{y} = a^2 + b^2.$$

86. The distance from the centre at which the norma cuts the axis of x, is, for the ellipse, $\frac{(a^2 - b^2)\,x}{a^2}$, and fo the hyperbola, $\frac{(a^2 + b^2)\,x}{a^2}$.

87. The *subnormal* is the portion of the axis which is intercepted between the ordinate and the normal. In the ellipse the subnormal $= -\frac{b^2x}{a^2}$, and in the hyperbola $= \frac{b^2x}{a^2}$

88. **Problem.** To find two points, S and H, in the axis major of an ellipse, equidistant from the centre, the *sum* of whose distances from a given point in the curve shall be equal to the axis major.

This distance AS or AH will be found to be $=\sqrt{(a^2-b^2)}$, a quantity independent of the position of the point in the curve, and therefore the same for *every* point in the curve.

89. The two points, S and H, found by the solution of the problem in the last Article, are called the *foci* of the ellipse.

90. **Problem.** To find two points, S and H, in the axis major of an hyperbola, or in the axis major produced, equidistant from the centre, the difference of whose distances from a given point in the curve shall be equal to the axis major.

This distance AS or AH will be found to be $=\sqrt{(a^2+b^2)}$, a quantity independent of the position of the point in the curve, and therefore the same for *every* point in the curve.

91. The two points, S and H, in the axis major produced, of an hyperbola, determined by the solution of the last problem, are called the *foci* of the hyperbola.

92. The ratio $\frac{AS}{AB}$ of the distance of the focus from the centre in the ellipse or hyperbola, to the semiaxis major, is called the *eccentricity*, and is usually denoted by e.

Since $ae=\sqrt{(a^2-b^2)}$ in the ellipse, and $ae=\sqrt{(a^2+b^2)}$ in the hyperbola, it follows that e is less than 1 in the ellipse and greater than 1 in the hyperbola.

93. The distance SP from the *focus* to any given point in the curve, is called the focal distance.

In the ellipse, the focal distances SP and HP are $a-ex$ and $a+ex$ respectively: in the hyperbola, SP and HP are $ex-a$ and $ex+a$ respectively.

94. The product of the focal distances is equal to the square of the semi-conjugate diameter: that is, $SP \times HP = AD^2$, if AD be the semi-conjugate diameter to AP.

95. The length of the perpendicular (Sy) from the focus (S) upon the tangent

$$= b\sqrt{\left(\frac{a - ex}{a + ex}\right)} = b\sqrt{\left(\frac{SP}{HP}\right)}:$$

and the length of the perpendicular (Hy') from the other focus (H) is $= b\sqrt{\left(\frac{HP}{SP}\right)}$.

96. The rectangle formed by the two perpendiculars (Sy, Hy') upon the tangent, is equal to (b^2) the square of the semiaxis minor.

97. The tangent makes equal angles with the focal distances. For

$$\frac{Sy}{SP} = \frac{Hy'}{HP} = \frac{b}{\sqrt{(SP \times HP)}} = \frac{b}{AD}.$$

This proposition furnishes a very easy geometrical construction for determining the position of the tangent, and therefore of the conjugate diameter.

98. The locus of the intersection of the perpendicular from either focus upon the tangent, with the tangent, is a circle described upon the axis major.

The propositions in the last five Articles are common both to the ellipse and hyperbola.

The following propositions belong to the hyperbola, considered with respect to its asymptotes.

99. The equations of the asymptotes are deduced from the equation of the hyperbola, by supposing x and therefore y to be infinitely great.

For the equation of the system of hyperbolas

$$a^2y^2 - b^2x^2 = \pm a^2b^2,$$

becomes $a^2y^2 - b^2x^2 = 0$, if x and y be infinitely great; which includes the equations

$$y + \frac{bx}{a} = 0 \text{ and } y - \frac{bx}{a} = 0,$$

of two lines or asymptotes passing through the origin of the co-ordinates, and making angles, whose tangents are $-\frac{b}{a}$ and $\frac{b}{a}$ respectively, with the axis of x.

100. To transform the equation of the hyperbola from ectangular to oblique co-ordinates coinciding with the asymp-otes. Make, since $\cos\theta = \frac{a}{\sqrt{(a^2+b^2)}} = -\cos\theta'$,

$$x = \frac{a}{\sqrt{a^2+b^2}}(x' - y'),$$

$$y = \frac{b}{\sqrt{a^2+b^2}}(x' + y').$$

nd the transformed equation becomes

$$x'y' = \frac{a^2+b^2}{4} \text{ for the hyperbola,}$$

$$-x'y' = \frac{a^2+b^2}{4} \text{ for the conjugate hyperbola.}$$

101. If $a = b$, the asymptotes, and therefore the axes f the system of hyperbolas referred to them, are rectangular: uch hyperbolas are called *rectangular* or *equilateral* hyperbolas.

102. All parallelograms included between the asymptotes nd the curve are equal to each other.

This property is merely the geometrical interpretation of he equation in Art. 99.

103. If a chord of the hyperbola be produced to meet the asymptote both ways, the parts of it intercepted between the curve and the asymptotes will be equal.

104. The tangent to the hyperbola at any point intersects the asymptote (considered as an axis) at twice the distance between the centre and the ordinate.

This property furnishes a very easy geometrical construction for determining the tangent.

SECT. 6.

Equation and properties of the Parabola.

105. It was found, in the discussion of the general equation of curves of the second order, that the terms involving the simple powers of x and y could not be exterminated *simultaneously*, when $4ac - b^2 = 0$, (Arts. 49. 50.); in this case, therefore, the curve can possess no *centre*, (Art. 55), and is consequently neither an ellipse nor an hyperbola: it remains to determine its nature and principal properties.

106. In the case under consideration, the general equation (a) is reducible to the form

$$(y\sqrt{a} + x\sqrt{c})^2 + dy + ex + f = 0 \ldots . (s).$$

107. The equation in the last Article (s) may be transformed into

$$Ay'^2 + Dy' + Ex' + f = 0 \ldots\ldots\ldots\ldots (t),$$

by moving the axes (without changing the origin) through an angle θ, where $\tan\theta = -\sqrt{\frac{c}{a}}$: in this case, $A = a + c$,

$$D = d\cos\theta - e\sin\theta = \frac{d\sqrt{a} + e\sqrt{c}}{\sqrt{(a+c)}},$$

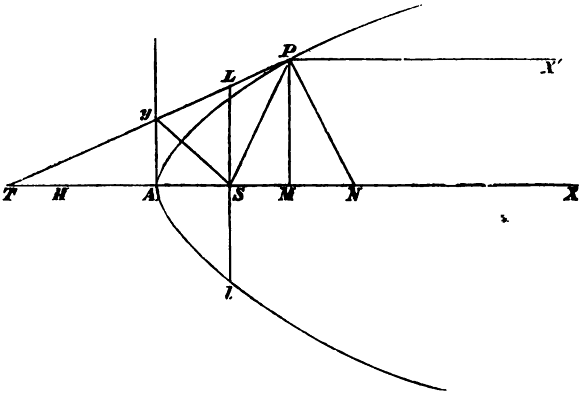
$$\text{and } E = d\sin\theta + e\cos\theta = -\frac{(d\sqrt{c} - e\sqrt{a})}{\sqrt{(a+c)}}.$$

108. If we move the *origin* of the co-ordinates (without changing the direction of the axes) to a point whose ordinate $= -\frac{D}{2A}$ and whose abscissa $= \frac{D^2}{4AE} - \frac{f}{E}$, we shall get the equation (making $a = -E$, and suppressing accents,)

$$y^2 = ax \ldots\ldots\ldots\ldots\ldots\ldots (u).$$

This is the most simple form which the equation (s) is capable of receiving.

It represents a curve, in which the axis of x bisects all the chords or double ordinates which are perpendicular to it:



it passes through the origin of the co-ordinates and forms two infinite branches upon the same side of the axis of y, and which are also symmetrical with respect to the axis of x: there is no part of the curve, *in the plane of xy*, corresponding to negative values of x: it is called the *parabola*.

109. Let the equation

$$y^2 = ax \ldots\ldots\ldots\ldots\ldots\ldots (u)$$

be transformed from rectangular to oblique axes, making the angles θ and θ' with the axis of x. (Art. 40.)

110. If $\theta = 0$, or if the axis of x' continue parallel to that of x, there is always a position of it in which it will continue to be a *diameter* of the parabola, whatever be the value of θ', or whatever be the angle of ordination of the axes or of the corresponding chords.

111. If $\theta = 0$, and if θ' be given, then the extremity of the corresponding diameter is a point in the curve, whose abscissa (with respect to the old axes) $= \frac{a}{4} \cot^2 \theta'$, and whose ordinate $= \frac{a}{2} \cot \theta'$.

112. The equation of the curve referred to the new axes is

$$y'^2 = \frac{a}{\sin^2 \theta'} \cdot x' \ldots\ldots\ldots\ldots\ldots (v).$$

113. The tangent to the parabola at the vertex of *any* diameter, is parallel to the corresponding system of chords, and its inclination to the principal axis, is therefore equal to θ'.

114. The equation of the tangent of the parabola, referred to the principal axis, is

$$y' = \frac{\sqrt{a}}{2\sqrt{x}} (x' + x).$$

115. If a negative distance from the vertex AT be taken upon the principal axis, which is equal to the abscissa AM of the point of contact, the tangent will pass through it.

This property affords a very easy geometrical construction of the tangent.

116. The equation of the normal of the parabola, referred to the principal axis, is

$$y' - y = -\frac{2y}{a}(x' - x).$$

117. The subnormal $MN = \frac{a}{2}$, and is therefore of the same length for every point of the curve.

118. PROBLEM. To find two points, S and H, in the principal axis, equidistant from the vertex A of the parabola, the distance of one of which, SP, from a given point P in the curve, is equal to the distance of the other, HM, from the foot of the ordinate.

The distance of this point from the vertex $= \frac{a}{4}$, which is independent of the co-ordinates of the given point in the curve, and which is therefore the same for every point in the curve: it is called the *focus* of the parabola.

119. The distance SP is called the focal distance: it is equal by the last Article to $x + \frac{a}{4}$.

120. The tangent to the curve at P makes equal angles with the focal distance and the axis.

If the tangent meets the axis in T, then $ST = SP$.

121. The perpendicular (Sy) let fall from the focus upon the tangent of the parabola $= \sqrt{(SP \times SA)}$, where SP is the focal distance of the point of contact, and SA is the distance between the vertex and the focus.

122. The locus of the intersection of the perpendicular upon the tangent and the tangent, for different points in the curve, is the straight line Ay, which is the tangent to the vertex of the parabola.

123. The *latus rectum* of the parabola (as well as of the ellipse and hyperbola) is the chord at right angles to the axis, which passes through the focus.

It is equal to a, or to four times the distance between the vertex and focus.

124. The *parameter* of any point of the parabola, is the chord passing through the focus, which corresponds to that diameter.

It is equal to $\frac{a}{\sin^2\theta'}$, or to four times the distance between the focus and the vertex of the diameter.

SECT. 7.

General properties of the three Curves

125. If a series of parallel lines intersect another series of parallel lines, and likewise cut any of the curves represented by the general equation (a), Art. 48, the rectangles contained by their segments will bear a constant ratio to one another.

The equation (a) transformed to a system of *axes* parallel to the two systems of parallel lines respectively, will become

$$a'y^2 + b'xy + c\,x^2 + d'y + e'x + f = 0:$$

and if the origin of the co-ordinates (the direction of the axes being unchanged) was conceived to be transferred successively to the different points of intersection of the parallel lines, no change would take place in the coefficients a', b', c': let P be one of those points, and Py, Py' the segments in the axis of y, and Px and Px' those in the axis of x: then we shall find

$$Py \times Py' = \frac{f'}{a'}: \text{ and } Px \times Px' = \frac{f'}{c'}:$$

$$\text{consequently, } \frac{Py \times Py'}{Px \times Px'} = \frac{c'}{a'},$$

a conclusion which is true for all positions of the point P.

126. If $a' = c'$, or if the curve be a circle, then

$$\frac{Py \times Py'}{Px \times Px'} = 1.$$

This is the general property of a circle demonstrated in *Euclid*, Book III. Prop. 35 and 36.

127. If Py and Px be parallel to the semiaxes b and a respectively, then $\frac{Py \times Py'}{Px \times Px'} = \frac{b^2}{a^2}$ for the ellipse, and $\frac{-b^2}{a^2}$ for the hyperbola: if they be parallel to a system of diameters, d and d', then $\frac{Py \times Py'}{Px \times Px'} = \frac{d^2}{d'^2}$ for the ellipse, and $\frac{-d^2}{d'^2}$ for the hyperbola.

128. If Px coincide with the axis of a parabola and if Py be perpendicular to it, then $Py = Py'$ and Px' is infinite: in this case, $c' = 0$ and $\frac{Py^2}{Px \times \infty} = \frac{o}{a'} = 0$: and if we consider $\infty \times o = a$,* (a finite quantity,) we shall get

$$Py^2 = a \times Px, \text{ or } y^2 = ax,$$

which is the equation of the parabola.

SECT. 8.

Recapitulation of the classes of Curves included in the general equation, with a notice of some excepted cases.

129. If $4ac - b^2$ be *positive:* the equation (a) expresses, *in the plane of* x, y,

(1) An ellipse, generally.

(2) A circle, if $a = c$, when the axes are rectangular; but if the axes be inclined to each other at an angle θ, when $a = c$, and when $b = 2a \cos \theta$.

(3) If $(bd - 2ae)^2$ be not greater than

$$(b^2 - 4ac)\,(d^2 - 4af),$$

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* For the sense in which such a conclusion may be justified, see *Algebra*, p. 280. Note.

no part of the curve is found in the plane of x, y: in this case, a positive quantity is always considered greater than a negative quantity.

(4) If $(bd - 2ae)^2 = (b^2 - 4ac)(d^2 - 4af)$, the portion of the curve in the plane of xy is a point, whose co-ordinates are

$$\frac{2ae - bd}{4ac - b^2} \text{ and } \frac{2cd - be}{4ac - b^2}.$$

130. If $4ac - b^2$ be *negative*, the equation (a) expresses,

(1) A system of hyperbolas generally.

(2) If $(bd - 2ae)^2 = (b^2 - 4ac)(d^2 - 4af)$, it represents two straight lines, whose equations are

$$y = -\frac{\{b + \sqrt{(b^2 - 4ac)}\}\,x}{2a} - \frac{d + \sqrt{(d^2 - 4af)}}{2a}$$

$$\text{and } y = -\frac{\{b - \sqrt{(b^2 - 4ac)}\}\,x}{2a} - \frac{d - \sqrt{(d^2 - 4af)}}{2a}$$

respectively: in this case, the equation (a) is the difference of two squares.

(3) An equilateral hyperbola, if $a = -c$, $b = 0$, the axes being rectangular.

131. If $b^2 - 4ac = 0$: the equation represents

(1) A parabola, generally.

(2) If $bd - 2ae = 0$, it represents two parallel lines, whose equations are

$$y = -\frac{bx}{2a} - \frac{d \pm \sqrt{(d^2 - 4af)}}{2a}:$$

if $d^2 = 4af$, these two lines degenerate into one; and if, under the same circumstances, d^2 be less than $4af$, no part of the curve is found in the plane of xy.

Sect. 9.

n the equations of the Ellipse, Hyperbola and Parabola, in terms of polar co-ordinates.

132. The position of a point is *determined* by its dis-ance from a given point or *pole*, and the angle which that istance makes with a given line or axis.

133. The distance of a point in a curve from its pole is called the *radius vector* or *polar distance:* and the angle which it makes with the assumed axis is called the angle of *revolution:* they are usually called (by a slight abuse of language) the *polar co-ordinates* of the point of the curve, to distinguish them from the *rectangular* or *oblique* linear co-ordinates, which are used for the same purpose.

We shall denote, in the subsequent articles, the radius vector by u, and the angle of revolution by θ.

134. Let the pole coincide with the origin of the *rectangular* co-ordinates, and let the assumed axis coincide with, or be parallel to, the axis of x.

Then $x = u \cos \theta$ and $y = u \sin \theta$; and the substitution of these values in the equation of the curve, will furnish us with its equation when referred to polar co-ordinates.

135. If the pole does not coincide with the origin of the rectangular co-ordinates, or if the assumed axis does not coincide with or be parallel to the axis of x, then the equation of the curve must be transformed in such a manner that the origin may coincide with the pole, and the assumed axes may coincide with or be parallel to the new axis of x.

136. If it be required to pass from polar to rectangular co-ordinates, having the pole for the origin, and the assumed axis coinciding with or parallel to the axis of x, we must make

$$\sqrt{(x^2+y^2)} = u \quad \text{and} \quad \cos\theta = \frac{x}{\sqrt{x^2+y^2}},$$

$$\text{and therefore } \sin\theta = \frac{y}{\sqrt{(x^2+y^2)}}.$$

137. **Problem.** To find the polar equation of the circle, the pole being the centre.

If we substitute $u\cos\theta$ for x and $u\sin\theta$ for y, in the equation

$$x^2 + y^2 = \gamma^2, \quad \text{(Art. 44.)}$$

we shall get $u^2 = \gamma^2$ or $u = \gamma$; a polar equation, in which u is independent of θ.

138. **Problem.** To find the polar equation of the ellipse, the pole being one of the foci, and the angle of revolution being reckoned from the axis major.

The equation

$$a^2y^2 + b^2x^2 = a^2b^2,$$

must be transformed into

$$a^2y^2 + b^2(ae - x')^2 = a^2b^2,$$

in order to transfer the origin from the centre to the focus or pole: if we make

$$y = u\sin\theta \quad \text{and} \quad x' = u\cos\theta,$$

we shall obtain, after proper reductions,

$$u = \frac{a(1-e^2)}{1 - e\cos\theta}.$$

The same result may be obtained more readily from the value of the radius vector SP, given in Art. 92: inasmuch as

$$SP = u = a - ex = a - e(ae - x')$$

$$= a - ae^2 + eu\cos\theta,$$

$$\text{we get } u = \frac{a(1-e^2)}{1 - e\cos\theta}.$$

139. **Problem.** To find the polar equation of the hyperbola.

$$u = \frac{a(e^2 - 1)}{1 - e \cos \theta}.$$

140. **Problem.** To find the polar equation of the parabola.

$$u = \frac{\frac{a}{2}}{1 - \cos \theta} = \frac{a}{4 \cos^2 \frac{\theta}{2}}.$$

141. **Proposition.** If a chord PSp be drawn through the focus S of an ellipse, hyperbola or parabola, meeting the curve in P and p, then

$$\frac{1}{SP} + \frac{1}{Sp} = \frac{2}{L},$$

where L is the *latus rectum* (Art. 123.) of the curve.

This conclusion follows immediately from the polar equations of the several curves.

Sect. 10.

On the Position of a Point in Space, when referred to rectangular or oblique axes, which are not in the same plane with it.

142. The position of a *point in space*, with respect to three given planes, may be determined by the three *perpendiculars* let fall upon them.

143. If each of these planes be at right angles to the other two, they are called *rectangular co-ordinate* planes, and the *perpendiculars* upon them are called *rectangular co-ordinates*.

144. The intersections of the rectangular co-ordinate planes with each other, are called *rectangular axes*, and are

usually termed the axes of x, y, z respectively, where x, y, z are reckoned upon them from their common origin, and are severally equal and parallel to the three rectangular co-ordinates.

145. The co-ordinate planes are usually termed the planes of xy, xz, yz respectively, from the denominations of the two axes which are common to each of them.

146. The position of a point would be equally determined, if referred to three *oblique* axes, (the intersections of three planes not at right angles to each other), to each of which its co-ordinates were respectively parallel: when, however, axes are mentioned, they are always considered to be rectangular, unless the contrary be expressed.

147. Examination of the signs of the co-ordinates of a point, in any one of the eight trihedral angles, which the co-ordinate planes form with each other.

148. If x, y, z be the co-ordinates of a point P, and if p', p'', p''' be the perpendiculars let fall from it upon the axes of x, y, z respectively, then

$$p' = \sqrt{(y^2+z^2)}, \quad p'' = \sqrt{(x^2+z^2)}, \quad p''' = \sqrt{(z^2+y^2)}.$$

149. If d be the distance of the point P from the origin of the co-ordinates x, y, z, then

$$d = \sqrt{(x^2+y^2+z^2)}.$$

150. If a, b, c be the angles which the distance d make with the axes of x, y, z respectively, then

$$x = d \cos a, \quad y = d \cos b, \quad z = d \cos c.$$

151. It follows from the two last Articles, that

$$\cos^2 a + \cos^2 b + \cos^2 c = 1.$$

152. If a', b', c' be the angles which the distance d makes with the planes of xy, xz, yz respectively, then

$$a' = \frac{\pi}{2} - c, \quad b' = \frac{\pi}{2} - b, \quad c' = \frac{\pi}{2} - a;$$

and therefore

$$\sin^2 a' + \sin^2 b' + \sin^2 c' = 1.$$

153. If α, β, γ and α', β', γ' be the co-ordinates of two given points, their distance from each other (δ)

$$= \sqrt{\{(\alpha - \alpha')^2 + (\beta - \beta')^2 + (\gamma - \gamma')^2\}}.$$

If A, B, C be the angles which (δ) makes with the axes of x, y, z respectively, then

$$\cos A = \frac{\alpha - \alpha'}{\delta}, \quad \cos B = \frac{\beta - \beta'}{\delta}, \quad \cos C = \frac{\gamma - \gamma'}{\delta}.$$

154. If the axes to which the point is referred be oblique, and if the angles between the axes of x and y, x and z, y and z, be θ, θ', θ'' respectively, then

$$d^2 = x^2 + y^2 + z^2 + 2xy \cos \theta + 2xz \cos \theta' + 2yz \cos \theta''.$$

This is the expression for the diagonal of a parallelopipedon, if x, y, z be the adjacent edges, and if θ, θ', θ'' be the angles between them.

155. If a, b, c be the angles which d makes with the axes of x, y and z respectively, then we shall find

$$d \cos a = x + y \cos \theta + z \cos \theta',$$

$$d \cos b = y + x \cos \theta + z \cos \theta'',$$

$$d \cos c = z + y \cos \theta'' + x \cos \theta'.$$

156. It follows from the two last Articles, that

$$d = x \cos a + y \cos b + z \cos c.$$

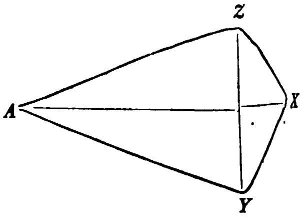
157. If X, Y, Z denote the angles between the plane of xy and xz, xy and yz, xz and yz, then we shall find

$$\cos X = \frac{\cos\theta'' - \cos\theta\cos\theta'}{\sin\theta\,\sin\theta'};$$

$$\cos Y = \frac{\cos\theta' - \cos\theta\cos\theta''}{\sin\theta\,\sin\theta''};$$

$$\cos Z = \frac{\cos\theta - \cos\theta'\cos\theta''}{\sin\theta'\,\sin\theta''}.$$

For, if AX, AY, AZ be the axes of x, y and z, and if XY and XZ be drawn at right angles to AX in the planes of xy and xz respectively, then, if $AX = d$, we have



$$XY = d\tan\theta, \qquad AY = d\sec\theta,$$
$$XZ = d\tan\theta', \qquad AZ = d\sec\theta',$$

and therefore

$$ZY^2 = AY^2 + AZ^2 - 2AY \times AZ \cos ZAY$$
$$= XY^2 + XZ^2 - 2XY \times XZ \cos X,$$

$$\text{or } d^2\sec^2\theta + d^2\sec^2\theta' - 2d^2\sec\theta\sec\theta'\cos\theta''$$
$$= d^2\tan^2\theta + d^2\tan^2\theta' - 2d^2\tan\theta\tan\theta'\cos X,$$

an equation, from which the value of $\cos X$ above given is very easily deduced.

This is a most important proposition, and constitutes the fundamental theorem in Spherical Trigonometry.

158. If $S = \dfrac{\theta + \theta' + \theta''}{2}$, we shall find

$$\sin X = \frac{\sqrt{(1 + 2\cos\theta\cos\theta'\cos\theta'' - \cos^2\theta - \cos^2\theta' - \cos^2\theta'')}}{\sin\theta\,\sin\theta'}$$

$$= \frac{2\sqrt{\{\sin S \,.\, \sin(S-\theta)\sin(S-\theta')\sin(S-\theta'')\}}}{\sin\theta\,\sin\theta'}$$

$$= \frac{N}{\sin\theta\,\sin\theta'},$$

$$\sin Y = \frac{N}{\sin\theta \sin\theta''},$$

$$\sin Z = \frac{N}{\sin\theta' \sin\theta''}.$$

159. If ϕ, ϕ', ϕ'' be the angles which the axes of x, y, z make with the planes of xy, xz and yz respectively, we shall find

$$\sin\phi = \frac{N}{\sin\theta},$$

$$\sin\phi' = \frac{N}{\sin\theta'};$$

$$\sin\phi'' = \frac{N}{\sin\theta''}.$$

160. The volume of the parallelopipedon, whose adjacent edges are x, y, z, making angles θ, θ', θ'' with each other, is

$$= xyz\sqrt{(1 + 2\cos\theta\cos\theta'\cos\theta'' - \cos^2\theta - \cos^2\theta' - \cos^2\theta'')}.$$

SECT. 11.

On the equation of a Straight Line in space when referred to rectangular or other axes not in the same plane with it.

161. The *orthographical* projection of a straight line upon a plane, is the line intercepted between two perpendiculars let fall from the extremities of the given line upon the given plane.

162. The position of a *straight line in space* may be determined from its *orthographical* projections upon the three co-ordinate planes, any one of which may be deduced from the other two.

163. If the equations of the projections of the straight line upon the planes xz and yz be

$$x = tz + s \,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\, (1),$$
$$y = t'z + s' \,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\, (2),$$

then we shall get, by eliminating z from the equations (and (2),

$$y = \frac{t'}{t}x + \frac{ts' - st'}{t} \dots\dots\dots\dots (3).$$

164. The equations to a line in space which passes through a point whose co-ordinates are α, β, γ, are

$$x - \alpha = t\,(z - \gamma),$$
$$y - \beta = t'\,(z - \gamma).$$

165. If the line passes through the origin of the co-ordinates, or if

$$\alpha = 0, \quad \beta = 0, \quad \gamma = 0,$$

then its equations become

$$x = tz \text{ and } y = t'z.$$

166. The equations of a line passing through two points whose co-ordinates are α, β, γ and α', β', γ', are

$$x - \alpha = \frac{\alpha' - \alpha}{\gamma' - \gamma}\,(z - \gamma),$$
$$y - \beta = \frac{\beta' - \beta}{\gamma' - \gamma}\,(z - \gamma).$$

167. If a, b, c be the angles which a line passing through the origin, whose equations are $x = tz$ and $y = t'z$, (or any line *parallel* to it, whose equations are $x = tz + s$ and $y = t'z + s'$), makes with the axes of x, y, z respectively, then

$$t = \frac{\cos a}{\cos c}, \quad t' = \frac{\cos b}{\cos c}.$$

For

$$\tan a = \frac{\sqrt{(z^2 + y^2)}}{x} = \frac{\sqrt{(1 + t'^2)}}{t} \text{ and } \tan c = \frac{\sqrt{(x^2 + y^2)}}{z} = \sqrt{t^2 + t'^2},$$

$$\text{and therefore } \frac{\cos a}{\cos c} = t:$$

and similarly for the value of t'.

168. It follows from the last Article, that

$$\cos a = \frac{t}{\sqrt{(1+t^2+t'^2)}}, \quad \cos b = \frac{t'}{\sqrt{(1+t^2+t'^2)}},$$

$$\cos c = \frac{1}{\sqrt{(1+t^2+t'^2)}}.$$

169. Two lines expressed by the systems of equations

$x = tz + s$, $y = t'z + s'$ and $x_{,} = t_{,}z_{,} + s_{,}$, $y_{,} = t_{,}'z + s_{,}'$,

will not meet, unless the equation

$$\frac{s - s_{,}}{t - t_{,}} = \frac{s' - s_{,}'}{t' - t_{,}'}$$

is satisfied.

170. If the conditions in the last Article be satisfied, the co-ordinates (α, β, γ) of the point of intersection are

$$\gamma = -\frac{s - s_{,}}{t - t_{,}}, \quad \beta = \frac{t's_{,}' - t_{,}'s'}{t' - t_{,}'}, \quad \alpha = \frac{ts_{,} - t_{,}s}{t - t_{,}}.$$

171. If θ be the angle of inclination to each other of two lines (whether they meet or not) which are expressed by the equations in Art. 163, then we shall find

$$\cos\theta = \frac{1 + tt_{,} + t't_{,}'}{\sqrt{(1+t^2+t_{,}^2)}\sqrt{(1+t'^2+t_{,}'^2)}}.$$

For if lines parallel to the given lines be drawn through the origin, and if x, y, z and x', y', z' be the co-ordinates of two points in those lines, d and d' their distances from the origin, and δ their distance from each other, then by Art. 149 and 153, we shall get

$$\cos\theta = \frac{d^2 + d'^2 - \delta^2}{2dd'}$$

$$= \frac{x^2+y^2+z^2+x'^2+y'^2+z'^2-(x-x')^2-(y-y')-(z-z')^2}{2dd'}$$

$$= \frac{xx' + yy' + zz'}{\sqrt{(x^2+y^2+z^2)}\sqrt{(x'^2+y'^2+z'^2)}}$$

$$= \frac{1 + tt_{,} + t't_{,}'}{\sqrt{(1+t^2+t_{,}^2)}\sqrt{(1+t'^2+t_{,}'^2)}}.$$

172. If a, b, c and a', b', c' be the angles which two lines make with the axes x, y, z respectively, then

$$\cos\theta = \cos a \cos a' + \cos b \cos b' + \cos c \cos c'.$$

This is deduced from Art. 168 and 171.

173. If a line be referred to oblique axes, its equations will retain the same form as when it is referred to rectangular axes.

The propositions contained in Art. 163, 164, 165, 166, 169, 170, are equally applicable to lines expressed by their equations, when referred to oblique as well as to rectangular axes.

SECT. 12.

On the Equation of a Plane.

174. DEFINITION. A plane is a surface, every point of which is at *equal* distances from two given points.

It follows from this definition, that the straight line which joins any two points in a plane surface, lies wholly in that surface.

175. The co-ordinates of a plane or of any other surface, are the co-ordinates of any point whatsoever in it: and the equation of a plane or of any other surface, is the equation which determines the relation between the co-ordinates, so that the points to which they correspond may always be found in the given surface.

176. A *plane* may be determined in *position*, by the length and position of the perpendicular let fall upon it from the origin of the co-ordinates.

177. If δ be the length of this perpendicular, α, β, γ the co-ordinates of the point where it meets the plane, and

x, y, z the co-ordinates of any point whatever in the plane, then

$$\alpha x + \beta y + \gamma z = \delta^2 \,\ldots\ldots\ldots\ldots (1),$$

which is the equation of the plane.

For if the perpendicular (δ) be extended to the length 2δ, to a point whose co-ordinates are 2α, 2β, 2γ, and which is at the *same* distance from *any* point whatever in the plane as the origin of the co-ordinates, then we get

$$x^2 + y^2 + z^2 = (2\alpha - x)^2 + (2\beta - y)^2 + (2\gamma - z)^2:$$

from whence the equation (1) is obtained.

178. If a, b, c be the angles which the perpendicular δ makes with the axes of x, y, z respectively, then

$$x \cos a + y \cos b + z \cos c = \delta \,\ldots\ldots (2).$$

179. If d, d', d'' be the distances from the origin, at which the plane cuts the axes of x, y, z respectively, then

$$\frac{x}{d} + \frac{y}{d'} + \frac{z}{d''} = 1 \,\ldots\ldots\ldots\ldots (3).$$

180. If we make

$$\frac{-\cos a}{\delta} = \frac{-1}{d} = A, \quad \frac{-\cos b}{\delta} = \frac{-1}{d'} = B, \quad \frac{-\cos c}{\delta} = \frac{-1}{d''} = C,$$

and consequently $\delta = \dfrac{1}{\sqrt{(A^2 + B^2 + C^2)}}$,

(Art. 151.), then the equation of the plane becomes

$$Ax + By + Cz + 1 = 0 \,\ldots\ldots\ldots (4).$$

This is the form under which the equation of the plane is most frequently exhibited.

181. For the sake of symmetry, the equation (4), in the last Article, is sometimes put under the form

$$Ax + By + Cz + D = 0 \,\ldots\ldots\ldots (5);$$

in which case it involves a superfluous constant. Under such circumstances, however, we generally suppose $A = -\cos a$, and therefore $B = -\cos b$, $C = -\cos c$ and $D = \delta$, where a, b, c are the angles made by the plane with the co-ordinate planes yz, xz, xy, being severally equal to those which δ makes with the axes of x, y, z: upon this hypothesis there is no superfluous constant, since

$$A^2 + B^2 + C^2 = 1.$$

182. If the plane passes through the origin, $\delta = 0$, and therefore $D = 0$, and the equation (5) becomes

$$Ax + By + Cz = 0 \ldots\ldots\ldots\ldots\ldots (6).$$

183. The lines of intersection of the plane whose equation is

$$Ax + By + Cz + 1 = 0,$$

with the planes of xz, yz and xy, or its *traces* upon those planes, are expressed by the equations

$$z = \frac{-A}{C} x - \frac{1}{C} \text{ or } Ax + Cz + 1 = 0,$$

$$z = \frac{-B}{C} y - \frac{1}{C} \text{ or } By + Cz + 1 = 0,$$

$$x = \frac{-B}{A} y - \frac{1}{A} \text{ or } Ax + By + 1 = 0.$$

184. The equation of a plane may be determined from the equations of its *traces* upon any two of the co-ordinate planes.

185. The equation of a plane which passes through a point whose co-ordinates are α, β, γ, is

$$A(x - \alpha) + B(x - \beta) + C(x - \gamma) = 0.$$

186. If a plane passes through three points whose co-ordinates are α, β, γ; α', β', γ'; α'', β'', γ''; the constants in its equation will be found to be

$$A = \beta(\gamma' - \gamma'') + \beta'(\gamma'' - \gamma) + \beta''(\gamma - \gamma'),$$
$$B = \gamma(a' - a'') + \gamma'(a'' - a) + \gamma''(a - a'),$$
$$C = a(\beta' - \beta'') + a'(\beta'' - \beta) + a''(\beta - \beta'),$$
$$D = a(\beta'\gamma'' - \beta''\gamma') + a'(\beta''\gamma - \beta'\gamma'') + a''(\beta\gamma' - \beta'\gamma).$$

187. The equations of a line perpendicular to a plane, whose equation is

$$Ax + By + Cz + 1 = 0,$$

and which passes through a point whose co-ordinates are a, β, γ, are

$$x - a = \frac{A}{C}(z - \gamma),$$

$$y - \beta = \frac{B}{C}(z - \gamma).$$

For the projections of the line and the *trace* of the plane upon any one of the co-ordinate planes, must be perpendicular to each other.

188. The length of the perpendicular (p) let fall from the point whose co-ordinates are a, β, γ, upon the plane whose equation is

$$Ax + By + Cz + D = 0,$$

$$\text{is } = -\frac{Aa + B\beta + C\gamma + D}{\sqrt{(A^2 + B^2 + C^2)}}.$$

189. If $a = 0$, $\beta = 0$, $\gamma = 0$; or if the origin be the point from which the perpendicular is drawn, then

$$p = \delta = \frac{-D}{\sqrt{(A^2 + B^2 + C^2)}} = D, \text{ if } \sqrt{(A^2 + B^2 + C^2)} = 1.$$

190. To find the equation of a plane which shall pass through a point whose co-ordinates are a, β and γ, and be perpendicular to a line whose equations are

$$x = tz + s, \qquad y = t'z + s'.$$

The equation required is

$$t(x-\alpha)+t'(y-\beta)+(z-\gamma)=0.$$

191. To find the angle (θ) which a line whose equations are

$$x = tz + s, \qquad y = t'z + s',$$

makes with the plane whose equation is

$$Ax + By + Cz + D = 0.$$

$$\text{Sin }\theta = \frac{At + Bt' + C}{\sqrt{(A^2+B^2+C^2)}\sqrt{(1+t^2+t'^2)}} = \frac{At+Bt'+C}{\sqrt{(1+t^2+t'^2)}},$$

$$\text{if } \sqrt{(A^2+B^2+C^2)} = 1.$$

The angle θ is the complement of the angle formed by the given line and a line perpendicular to the plane: the preceding expression is immediately deduced therefore from Art. 190 and 171.

192. If θ be the angle of inclination to each other of two planes, whose equations are

$$Ax + By + Cz + D = 0,$$

$$A'x + B'y + C'z + D' = 0.$$

$$\text{Then } \cos\theta = \frac{AA' + BB' + CC'}{\sqrt{(A^2+B^2+C^2)}\sqrt{(A'^2+B'^2+C'^2)}}$$

$$= AA' + BB' + CC',$$

$$\text{if } A^2 + B^2 + C^2 = 1 \text{ and } A'^2 + B'^2 + C'^2 = 1.$$

The angle θ is the angle between the two perpendiculars let fall from the origin of the co-ordinates upon the two planes.

193. If the planes be at right angles to each other, then

$$AA' + BB' + CC' = 0.$$

194. If the second plane be the plane of xy, or parallel to it, then

$$\cos\theta = \frac{C}{\sqrt{(A^2 + B^2 + C^2)}} = \cos c,$$

where c is the angle formed by δ with the axis of z.

In a similar manner, we shall find

$$\cos b = \frac{B}{\sqrt{(A^2 + B^2 + C^2)}} \text{ and } \cos a = \frac{A}{\sqrt{(A^2 + B^2 + C^2)}}.$$

195. If therefore a, b, c and a', b', c' be the angles formed by the planes severally with the co-ordinate planes, we shall get

$$\cos\theta = \cos a \cos a' + \cos b \cos b' + \cos c \cos c'.$$

196. If the axes to which the plane is referred be oblique, and if a, b, c be the angles which the perpendicular (δ) from the origin let fall upon it, makes with the axes of x, y, z respectively, then the equation of the plane is

$$x \cos a + y \cos b + z \cos c = \delta.$$

This equation is identical in form and interpretation with the equation (Art. 178.) of the plane referred to rectangular axes.

The propositions contained in Art. 179, 180, 181, 182, 183, 184, 185, 186, 187, are equally applicable when the plane is referred to oblique and to rectangular axes.

197. If the co-ordinates of the point where the perpendicular meets the plane be α, β, γ, and if θ, θ', θ'' be the angles which the axes of x, y, z make with each other, then

$$(\alpha + \beta \cos\theta + \gamma \cos\theta')\, x + (\beta + \alpha \cos\theta + \gamma \cos\theta'')\, y$$
$$+ (\gamma + \beta \cos\theta'' + \alpha \cos\theta')\, z = \delta^2.$$

This equation becomes identical with the equation in Art. 177, when the axes become rectangular.

Sect. 13.

On the Transformation of Co-ordinates in space, the axes being always rectangular.

198. Let the origin of the co-ordinates *alone* be changed, the new co-ordinate planes remaining parallel to the old.

If α, β, γ be the co-ordinates of the new origin, then

$$x = x' + \alpha, \quad y = y' + \beta, \quad z = z' + \gamma.$$

199. Let the origin of the co-ordinates be *not* changed, the *new* co-ordinate planes being no longer parallel to the primitive.

We may consider the primitive co-ordinate planes of xy, xz, yz, as referred to the new co-ordinate planes of $x'y'$, $x'z'$, $y'z'$, by means of the equations (Art. 182.)

$$\left.\begin{aligned} Ax' + By' + Cz' &= 0 \\ A'x' + B'y' + C'z' &= 0 \\ A''x' + B''y' + C''z' &= 0 \end{aligned}\right\}\ldots\ldots\ldots\ldots(1),$$

where x', y', z' are the co-ordinates of a point in the primitive planes referred to the new.

Of the nine constants included in equations (1), three only are indeterminate: for (Art. 181.)

$$\left.\begin{aligned} A^2 + B^2 + C^2 &= 1 \\ A'^2 + B'^2 + C'^2 &= 1 \\ A''^2 + B''^2 + C''^2 &= 1 \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(2);$$

and since the planes are perpendicular to each other, we have also (Art. 193.)

$$\left.\begin{aligned} AA' + BB' + CC' &= 0 \\ AA'' + BB'' + CC'' &= 0 \\ A'A'' + B'B'' + C'C'' &= 0 \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

The other three constants must be determined from the specific *data*, which determine the position of the new co-ordinate planes with respect to the primitive.

If x, y, z be the co-ordinates of a point with respect to the primitive planes, and x', y', z' the co-ordinates of the same point with respect to the new, then we shall find

$$x = -Ax' - By' - Cz',$$
$$y = -A'x' - B'y' - C'z',$$
$$z = -A''x' - B''y' - C''z':$$

for x, y, z are the lengths of the perpendiculars let fall from the point upon the primitive planes. (Art. 188.)

The substitution, therefore, of these values of x, y, z, will effect the transformation required.

200. The *specific data*, employed in the determination of the coefficients of x', y', z', are of very various kinds, and generally speaking, lead to very complicated and embarrassing expressions: there is one form under which they may very generally be presented, which will enable us to determine them by means of three successive transformations of two co-ordinates, such as are given in Art. 35.

For let the *specific* data be the angle θ between the planes of xy and $x'y'$, and the angles ψ and ϕ made by their common intersection with the axes of x and x' respectively; and let us make the following transformations.

1st. The axis of z remaining fixed, let the axis of x revolve in the plane xy, through an angle ψ, until it reaches the common intersection of the planes xy and $x'y'$: calling $x_{,}$ and $y_{,}$ the new co-ordinates in the plane xy, we get (Art. 35.)

$$\left.\begin{aligned} x &= x_{,} \cos\psi - y_{,} \sin\psi \\ y &= x_{,} \sin\psi + y_{,} \cos\psi \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(1):$$

z is not changed.

2d. The axis of $x_,$ remaining fixed, let the plane xy revolve round it until it coincides with the plane $x'y'$: in other words, let the axes of z and $y_,$ revolve in their own plane, through the angle θ: the new values of z and $y_,$ being denoted by z' and y_2, we get

$$\left.\begin{array}{l} y_, = y_2 \cos\theta - z' \sin\theta \\ z' = y_2 \sin\theta + z \cos\theta \end{array}\right\} \dots\dots\dots\dots\dots\dots (2),$$

$x_,$ is not changed.

3d. The axis of z' remaining fixed, let the axis of $x_,$ move in the plane $x'y'$, through an angle ϕ, until it reaches the axis of x'; the new values of $x_,$ and y_2 being x' and y', we get

$$\left.\begin{array}{l} x_, = x' \cos\phi - y' \sin\phi \\ y_2 = x' \sin\phi + y' \cos\phi \end{array}\right\} \dots\dots\dots\dots\dots (3),$$

z' is not changed.

If we now substitute in equation (1) the values of $x_,$, $y_,$ and y_2, we shall finally get

$$\begin{aligned} x = {} & x' (\cos\psi \cos\phi - \cos\theta \sin\psi \sin\phi) \\ & - y' (\cos\psi \sin\phi + \cos\theta \sin\psi \cos\phi) \\ & + z' \sin\theta \sin\psi. \\ y = {} & x' (\sin\psi \cos\phi + \cos\theta \cos\psi \sin\phi) \\ & - y' (\sin\psi \sin\phi - \cos\theta \cos\psi \cos\phi) \\ & - z' \sin\theta \cos\psi. \\ z = {} & x' \sin\theta \sin\phi \\ & + y' \sin\theta \cos\phi \\ & + z' \cos\theta. \end{aligned}$$

201. When it is requisite to change both the origin of the co-ordinates and the direction of the rectangular axes, it is generally most convenient to effect the transformations successively; they may, however, be effected simultaneously, by making

$$\begin{aligned} x &= \alpha - Ax' - By' - Cz', \\ y &= \beta - A'x' - B'y' - C'z', \\ z &= \gamma - A''x' - B''y' - C''z'; \end{aligned}$$

replacing the coefficients of x', y', z', by their values determined in the last Article, or in any other manner, from the requisite data.

202. The dimensions of the equation of a curve formed by the intersection of a given plane with a curve surface, can never exceed the dimensions of the equation of the surface itself.

For the equation which results from the elimination of an unknown quantity between two equations, one of which is a simple equation, can never exceed the dimensions of the other equation.

203. To find the equation of the curve which is formed by the intersection of a given plane, with a surface whose equation is given.

Find the distance (d) of the intersection of a plane with the axis of x, and transform the equation of the surface so that the new origin may coincide with that point: or this transformation may be effected simultaneously with the one that follows.

Transform the equation of the surface to new axes, making the new axis of x' (when possible) coincide with the common intersection of the planes, so that the angle ϕ may be zero.

Make z' equal to zero in the transformed equation, and the resulting equation is the equation of the curve of section.

The expressions for x, y, z in the last Article, become, when $z'=0$ and $\phi=0$,

$$\begin{aligned} x &= x' \cos\psi - y' \cos\theta \sin\psi, \\ y &= x' \sin\psi + y' \cos\theta \cos\psi, \\ z &= y' \sin\theta. \end{aligned}$$

If $\psi = 90^0$, or if the common intersection of the planes coincide with the plane of yz, and the axis of y becomes that of x', then the preceding expressions become

$$\begin{aligned} x &= -y' \cos\theta, \\ y &= x', \\ z &= y' \sin\theta. \end{aligned}$$

If $\psi=0$, or if the axis of x' coincides with that of x, these expressions become

$$x=x',$$
$$y=y'\cos\theta,$$
$$z=y'\sin\theta.$$

SECT. 14.

On the Equation of the Sphere and its Sections.

204. A sphere is a curve surface, every point of which is equidistant from a given point, which is called the *centre.*

205. If α, β, γ be the co-ordinates of the centre, r the radius, and x, y, z the co-ordinates of any point in the surface, then

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=r^2\ldots\ldots(1).$$

206. If the origin of the co-ordinates be the centre, or if $\alpha=0$, $\beta=0$, $\gamma=0$, then the equation becomes

$$x^2+y^2+z^2=r^2\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

207. If the origin be the centre, the equation of the sphere is the same for all positions of the rectangular axes.

208. Every section of a sphere made by a plane is a circle.

Let the inclination of the cutting plane to the plane of xy be θ; and let us assume the common section of these planes to be *parallel* to the axis of x, and *coincident* with that of x': let d be the distance at which it cuts the axis of y: if we now make

$$x=x',$$
$$y=d+y'\cos\theta-z'\sin\theta,$$
$$\quad=d+y'\cos\theta,\ \text{when } z'=0;$$
$$z=y'\sin\theta+z'\cos\theta,$$
$$\quad=y'\sin\theta,\ \text{when } z'=0,$$

we shall get

$$x'^2 + y'^2 \sin^2\theta + (d + y' \cos\theta)^2 = r^2,$$

$$\text{or } x'^2 + y'^2 + 2d \cos\theta\, y' = r^2 - d^2,$$

the equation of a circle, whose radius is $= \sqrt{(r^2 - d^2 \sin^2\theta)}$.

209. The equation of a *tangent* plane to a point whose co-ordinates are x, y, z, is

$$xx' + yy' + zz' = r^2,$$

where x', y', z' are the co-ordinates of any point in the plane.

210. If the sphere be referred to oblique axes, where the angles between the axes of x and y, x and z, y and z, are θ, θ', θ'' respectively, its equation will be (Art. 154.)

$$x^2 + y^2 + z^2 + 2xy \cos\theta + 2xz \cos\theta' + 2yz \cos\theta'' = r^2.$$

211. The equation to a tangent plane to a point in the sphere whose oblique co-ordinates are x, y, z, is

$$(x + y\cos\theta + z\cos\theta')\, x' + (y + x\cos\theta + z\cos\theta'')\, y'$$
$$+ (z + y\cos\theta'' + x\cos\theta')\, z' = r^2.$$

For if r be the perpendicular upon a plane making angles a, b, c with the axes of x, y, z, x', y', z' being the co-ordinates of any point in the plane, its equation will be (Art. 156.)

$$r = x' \cos a + y' \cos b + z' \cos c.$$

Let the oblique co-ordinates of the point of intersection of the perpendicular and the plane be x, y, z, and let $x_{,}, y_{,}, z_{,}$ be the parts of the axes intercepted by perpendiculars let fall upon the axes of x, y, z from the point of intersection: then, because

$$\cos a = \frac{x_{,}}{r}, \quad \cos b = \frac{y_{,}}{r}, \quad \cos c = \frac{z_{,}}{r},$$

the equation of the plane becomes

$$x_{,}x' + y_{,}y' + z_{,}z' = r^2;$$

$$\begin{aligned} \text{but } x &= x + y\cos\theta + z\cos\theta', \\ y_{,} &= y + z\cos\theta + z\cos\theta'', \\ z &= z + y\cos\theta'' + x\cos\theta', \end{aligned}$$

the substitution of which gives the equation required (Art. 155).

SECT. 15.

On the Equations of a Cone and its Sections.

212. A *cone* is generated by a straight line, which passes always through a given point and describes by its extremity a given circle, which is called its base.

213. More generally, the base of a conical surface may be assumed to be any given curve.

214. As the generating line may be produced both ways, it will generate two opposite conical surfaces or *sheets* joined by their vertices, which will be equally expressed by the equation of the cone.

215. The *axis* of a cone is the line which joins the centre of its base and the vertex.

216. The cone will be *right* or *oblique*, according as its axis is at right angles or not to the plane of its base.

217. To find the equation of the right cone.

Let the centre of the base be the origin of the co-ordinates, and the axis of the cone the axis of z: let the length of the axis $= c$, and the co-ordinates of the fixed point in the generating line are 0, 0, c, and its equations

$$x = A(z-c), \text{ and therefore } A = \frac{x}{z-c},$$

$$y = B(z-c), \text{ and therefore } B = \frac{y}{z-c}:$$

if the equation of the base be $x_{,}^2+y_{,}^2=r^2$; then, when $z=0$, we must have $x=x_{,}=-Ac$ and $y=y_{,}=-Bc$: consequently,

$$x_{,}^2+y_{,}^2=A^2c^2+B^2c^2=r^2,$$

$$\text{or } \frac{c^2x^2}{(z-c)^2}+\frac{c^2y^2}{(z-c)^2}=r^2,$$

$$\text{or } x^2+y^2=\frac{r^2}{c^2}(z-c)^2,$$

which is the equation required.

218. To find the curve of section of a right cone.

The equation of the right cone

$$x^2+y^2=\frac{r^2}{c^2}(z-c)^2,$$

will remain unaltered, whatever be the position of the axes of x and y, in the plane of xy: let us suppose the plane to cut the cone at the distance c from the vertex, and making an angle θ with the plane of xy, and supposing the axes of x and x' to coincide with the line of section of the cutting plane, we shall find

$$x=x',$$

$$y=y'\cos\theta-z'\sin\theta=y'\cos\theta, \text{ when } z'=0,$$

$$\text{and } z=y'\sin\theta+z'\cos\theta=y'\sin\theta, \text{ when } z'=0:$$

consequently, the equation of the curve of section is

$$x'^2+y'^2\cos^2\theta=\frac{r^2}{c^2}(y'\sin\theta-c)^2,$$

$$\text{or } (c^2\cos\theta-r^2\sin\theta)\,y'^2+c^2x'^2+2cr^2\sin\theta\,y'-r^2c^2=0.$$

If $c\cos\theta$ be greater than $r\sin\theta$, or if $\tan\theta$ be less than $\frac{c}{r}$, or if the angle of inclination of the cutting plane to the base be less than the angle of inclination of the generating line to the axis, the curve of section is an *ellipse*.

If $c\cos\theta=r\sin\theta$, or if the cutting plane be parallel to the generating line, then the curve of section is a *parabola*.

If $c\cos\theta$ be less than $r\sin\theta$, or if the cutting plane be more inclined to the base than the generating line, then it must meet both *sheets* of the cone, and the curve of section is an *hyperbola*.

219. To find the equation of the oblique cone.

Let the centre of the base be the origin, the plane of xy coinciding with the base: and let a, b, c be the co-ordinates of the fixed point of the generating line, its equations being

$$x-a=A(z-c),$$
$$y-b=B(z-c);$$

then the equation required is

$$(az-cx)^2+(bz-cy)^2=r^2(z-c)^2.$$

Sect. 16.

Discussion of the general Equation of Curve Surfaces of the second Order.

220. The most general equation of curve surfaces of the second order is

$$ax^2+by^2+cz^2+2a'xy+2b'xz+2c'yz+2a''x+2b''y+2c''z$$
$$+d=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(a).$$

221. The terms involving the simple powers of x, y, z, may be exterminated from the equation (a), by changing the origin of the co-ordinates, in all cases, unless

$$abc-aa'^2-bb'^2-cc'^2+2a'b'c'=0.$$

The co-ordinates of the new origin are

$$\alpha=\frac{a''(a'^2-bc)+b''(cc'-a'b')+c''(bb'-a'c')}{abc-aa'^2-bb'^2-cc'^2+2a'b'c'},$$

$$\beta=\frac{b''(b'^2-ac)+c''(aa'-b'c')+a''(cc'-a'b')}{abc-aa'^2-bb'^2-cc'^2+2a'b'c'},$$

$$\gamma=\frac{c''(c'^2-ab)+a''(bb'-a'c')+b''(aa'-b'c')}{abc-aa'^2-bb'^2-cc'^2+2a'b'c'}.$$

The transformed equation is (suppressing the accents of x, y, z,)

$$ax^2 + by^2 + cz^2 + 2a'xy + 2b'xz + 2c'yz + d' = 0 \ldots\ldots\ldots (b),$$

the coefficients (as far as they go) being the same as in equation (a).

222. The origin of the co-ordinates in equation (b) is a *centre*, or a point in which all straight lines being drawn to meet the surface both ways, are bisected.

The surface in the case excepted in Art. 219, has no centre.

223. The terms in equation (b) which involve the products xy, xz, yz, may be exterminated by a change of the co-ordinate planes, the origin remaining unchanged: the equation is thus reduced to (suppressing accents)

$$Ax^2 + By^2 + Cz^2 + D = 0 \ldots\ldots\ldots\ldots\ldots (c).$$

This may be effected by the formulæ in Art. 203: and it may be shewn that there are always possible values of ψ and θ, which will cause the coefficients of the products xy, xz, yz, to vanish.

The same conclusion may be inferred from the existence of diametral planes, as shewn in the following Articles.

224. If a series of chords parallel to the line, whose equations are $x = tz$, $y = t_{,}z$ be drawn, the locus of their middle points is a plane, whose equation is

$$(at + c't_{,} + b')\,x' + (c't + bt_{,} + a')\,y' + (b't + a't_{,} + c)\,z' + a''t$$
$$+ b''t + c'' = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where x', y', z' are the co-ordinates of a point in the plane.*

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* *Hamilton's Analytical Geometry*, Art. 353 and 354.

If the equations of two other lines which determine the direction of the parallel chords be

$$x = t'z, \quad y = t_{,}'z, \quad \text{and} \quad x = t''z, \quad y = t_{,}''z,$$

the equations of the corresponding diametral planes are

$$(at' + c't_{,}' + b')x' + (c't' + bt_{,}' + a')y' + (b't' + a't_{,}' + c)z' + a''t' + b''t_{,}' + c'' = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$(at'' + c't_{,}'' + b')x' + (c't'' + bt_{,}'' + a')y' + (b't'' + a_{,}'' + c)z + a''t'' + b''t'' + c'' = 0 \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

225. The three planes whose equations are given in the last Article, will form a *system* of diametral planes, if the system of chords bisected by any one of them be parallel to the common intersection of the other two: this will take place when the following conditions are satisfied:*

$$b'(t + t') + a'(t_{,} + t_{,}') + c'(tt_{,}' + t_{,}t') + att' + bt_{,}t_{,}' + c = 0 \ldots (4),$$

$$b'(t + t'') + a'(t_{,} + t_{,}'') + c'(tt_{,}'' + t_{,}t'') + att'' + bt_{,}t_{,}'' + c = 0 \ldots (5),$$

$$b'(t' + t'') + a'(t_{,}' + t_{,}'') + c'(t't_{,}'' + t_{,}'t'') + att'' + bt_{,}'t_{,}'' + c = 0 \ldots (6).$$

226. Such diametral planes are said to be *conjugate* to each other, and the lines of intersection are called *conjugate* diameters.

227. The equations of the conjugate diameters are

$$x = tx \text{ and } y = t_{,}z, \quad x = t'z \text{ and } y = t_{,}'z, \quad x = t''z \text{ and } y = t''z:$$

there are only three equations (4), (5), (6), to determine the six unknown quantities

$$t \text{ and } t_{,}, \quad t' \text{ and } t_{,}', \quad t'' \text{ and } t_{,}'',$$

so that the three diameters corresponding to these may form a system of conjugate diameters: there are, therefore, an infinite number of such systems of conjugate diameters.

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* *Hamilton's Analytical Geometry*, Art. 355.

228. There is only *one* system of conjugate diameters, or of *axes*, which are at right angles to each other.

In this case, the three equations (Art. 193.)

$$1 + tt' + t_{,}t_{,}' = 0 \ldots\ldots\ldots\ldots\ldots\ldots(7),$$

$$1 + tt'' + t_{,}t_{,}'' = 0 \ldots\ldots\ldots\ldots\ldots\ldots(8),$$

$$1 + t't'' + t_{,}'t_{,}'' = 0 \ldots\ldots\ldots\ldots\ldots\ldots(9),$$

must be combined with the three equations (4), (5), (6), (Art. 225.): and they suffice therefore to determine all the six quantities t, t', t'', $t_{,}$, $t_{,}'$, $t_{,}''$.

229. If the co-ordinates are reckoned from the *centre* (in those surfaces which possess a centre), and be parallel to any system of *conjugate* diameters, the equation (*a*) will assume the form

$$Ax^2 + By^2 + Cz^2 + D = 0 \ldots\ldots\ldots\ldots(d).$$

230. The equation to the tangent plane to a point, (whose co-ordinates are x, y, z), in the surface whose equation is (*d*), is

$$Axx' + Byy' + Czz' + D = 0.$$

For the tangent plane may be easily shewn to be necessarily parallel to the plane which is *conjugate* to the *diameter* passing through the point of contact: the equation of the *conjugate* plane is, making $a = A$, $b = B$, $c = C$, $a' = 0$, $b' = 0$, $c' = 0$, $a'' = 0$, $b'' = 0$, $c'' = 0$, (Art. 224.)

$$Atx_{,} + Bt_{,}y_{,} + Cz_{,} = 0;$$

or replacing t by $\frac{x}{z}$ and $t_{,}$ by $\frac{y}{z}$,

$$Axx_{,} + Byy_{,} + Czz_{,} = 0 \ldots\ldots\ldots\ldots\ldots(1).$$

If we now replace $x_{,}$ by $x' - x$, $y_{,}$ by $y' - y$, $z_{,}$ by $z' - z$, the equation of the plane passing through the point of contact *parallel* to the conjugate plane whose equation is (1), will be (Art. 185.)

$$Ax(x' - x) + By(y' - y) + Cz(z' - z) = 0,$$

$$\text{or } Axx' + Byy' + Czz' + D = 0 \ldots\ldots\ldots(2).$$

231. The equations of the normal, or of the line perpendicular to the tangent plane at the point of contact, are

$$x' - x = \frac{Ax}{Cz}(z' - z),$$

$$y' - y = \frac{By}{Cz}(z' - z).$$

232. If $2a'$, $2b'$, $2c'$ represent the lengths of the three conjugate diameters, then

$$a'^2 = -\frac{D}{A}, \quad b'^2 = -\frac{D}{B}, \quad c'^2 = -\frac{D}{C};$$

and the equation (*d*), Art. 229, becomes

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} + \frac{z^2}{c'^2} = 1 \ldots\ldots\ldots\ldots\ldots (e).$$

If any one or more of the quantities $\frac{-D}{A}$, $\frac{-D}{B}$, $\frac{-D}{C}$, should be negative, the corresponding sign of a'^2, b'^2, c'^2 must be changed from + to −: an attention to those changes of sign will enable us to distribute the curve surfaces which the equation (*d*) represents into their several classes.

If $2a$, $2b$, $2c$ represent the three rectangular or principal axes, then the equation (*d*), when the curve surface is referred to such axes, becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \ldots\ldots\ldots (f).$$

Any one or more of these terms may change their signs under the circumstances noticed above.

233. Given any system of conjugate diameters inclined at known angles to each other, to find the magnitude and position of the principal axes.

A principal diameter is a normal to the tangent plane at its extremity: let its length be $2r$, and the co-ordinates of one of its extremities α, β, γ: then its equations are (Art. 166.)

$$x = tz = \frac{\alpha}{\gamma}z \quad \text{and} \quad y = t_{,}z = \frac{\beta}{\gamma}z:$$

and the equation of the tangent plane corresponding is (Art. 230, 232.)

$$\frac{\alpha x'}{a'^2} + \frac{\beta y'}{b'^2} + \frac{\gamma z'}{c'^2} = 1 \ldots\ldots\ldots\ldots\ldots (1),$$

when referred to the conjugate diameters a', b', c' as axes. The equation of the same tangent plane considered as that of a sphere whose radius is r, referred to the axes a', b', c', making angles θ, θ', θ'' with each other, is (Art. 211),

$$\frac{x'}{r^2}(\alpha + \beta\cos\theta + \gamma\cos\theta') + \frac{y'}{r^2}(\beta + \alpha\cos\theta + \gamma\cos\theta'')$$
$$+ \frac{z'}{r^2}(\gamma + \beta\cos\theta'' + \alpha\cos\theta') = 1 \ldots\ldots\ldots (2).$$

If we now equate corresponding coefficients of (1) and (2), making $t = \frac{\alpha}{\gamma}$ and $t_{,} = \frac{\beta}{\gamma}$, we get

$$\left.\begin{aligned} t + t_{,}\cos\theta + \cos\theta' &= \frac{tr^2}{a'^2} \\ t_{,} + t\cos\theta + \cos\theta'' &= \frac{t'r^2}{b'^2} \\ 1 + t_{,}\cos\theta'' + t\cos\theta' &= \frac{r^2}{c'^2} \end{aligned}\right\} \ldots\ldots\ldots (3);$$

and if we eliminate t and $t_{,}$ from these equations, we get the equation

$$r^6 - (a'^2 + b'^2 + c'^2)r^4 + (a'^2b'^2\sin^2\theta + a'^2c'^2\sin^2\theta' + b'^2c'^2\sin^2\theta'')r^2$$
$$- a'^2b'^2c'^2(1 + 2\cos\theta\cos\theta'\cos\theta'' - \cos^2\theta - \cos^2\theta' - \cos^2\theta'') = 0.$$

The three values of r^2 derived from the solution of this cubic equation (which are real), are a^2, b^2, c^2, the squares of the principal axes of the curve surface.

The values of r^2 being known, the values of t and $t_{,}$ corresponding, which determine the position r, may be deduced from the equations (3).

234. $a'^2 + b'^2 + c'^2 = a^2 + b^2 + c^2;$

or the sum of the squares of any three conjugate diameters, is equal to the sum of the squares of the three principal axes.

If b'^2 or c'^2 or both of them be considered negative, then b^2 or c^2 or both of them must be considered negative likewise.

235. $a'^2b'^2c'^2(1+2\cos\theta\cos\theta'\cos\theta''-\cos^2\theta-\cos^2\theta'-\cos^2\theta'')$
$$=a^2b^2c^2;$$

or the volume of the parallelopiped constructed upon any system of conjugate diameters, is equal to that upon the principal axes.

For the square root of the first of these expressions has been shewn (Art. 196.) to be the volume of the parallelopiped constructed upon the edges a', b', c', where θ is the angle between a' and b', θ' the angle between a' and c', and θ'' the angle between b' and c'.

236. Distribution of the curve surfaces (possessing centres) which are included in the equation

$$Ax^2+By^2+Cz^2+D=0\ldots\ldots\ldots\ldots\ldots(d),$$

into their different classes.

237. If A, B, C, D have the same sign, the values of x, y, z, one or more of them involve $\sqrt{-1}$; and there is no curve surface corresponding.

238. If A, B, C have the same sign which is different from that of D, the equation (d) (referred to its principal axes) becomes $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$, and the surface corresponding is called the *ellipsoid*.

239. If A and B have the same sign which is different from that of C and D, the equation becomes

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1;$$

and the surface corresponding is called the *hyperboloid of one sheet* or *surface*.

240. If A has a different sign from that of B, C and D, the equation becomes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1;$$

and the surface corresponding is called the *hyperboloid of two sheets* or *surfaces*.

241. The *ellipsoid* is a surface which surrounds the centre at a *finite* distance in every direction.

242. The *principal* sections of the ellipsoid, or those made by the planes of xy, xz, yz are ellipses, whose semi-axes are a and b, a and c, b and c respectively.

243. All *parallel* sections of an ellipsoid are *similar* ellipses, or ellipses whose axes are in the same ratio to each other.

244. The section of an ellipsoid which passes through the *mean* axis $2b$, (or that which is intermediate in value between $2a$ and $2c$), and which makes with the plane of xy an angle, whose tangent $= \frac{c}{a}\sqrt{\left(\frac{a^2 - b^2}{b^2 - c^2}\right)}$, is a circle; and the same is therefore the case with all sections which are parallel to it.

245. It appears from the last Article, that an ellipsoid may be generated by a circle of variable radius moving parallel to itself, and whose centre describes a diameter in the principal section passing through the axes $2a$ and $2c$, and whose plane makes with the axis $2a$ an angle $= \tan^{-1}\frac{c}{a}\sqrt{\left(\frac{a^2 - b^2}{b^2 - c^2}\right)}$.

246. An ellipsoid has *six summits* or *vertices* at the extremities of the three principal axes.

247. If $a=b$, then the equation of the surface becomes

$$x^2+y^2+\frac{a^2z^2}{c^2}=a^2;$$

and all its sections parallel to the plane of xy become circles: in this case, the *ellipsoid* degenerates into a surface of revolution with two vertices only, and is called a *spheroid:* it is an *oblate* or *prolate* spheroid, according as a is greater or less than c.

248. If $a=b=c$, the *ellipsoid* becomes a *sphere.*

249. If c be *infinite*, the equation becomes

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1;$$

and if considered as the equation of a surface, it is that of *a cylinder* constructed upon an elliptical basis, whose semiaxes are a and b.

250. The *hyperboloid of one sheet* is an infinite surface, whose sections parallel to the plane of xy are *ellipses*, whilst those parallel to the planes of xz and yz are *hyperbolas:* all parallel sections are similar to the sections which pass through the centre.

251. The section of this hyperboloid which passes through the greater axis $2a$, and makes an angle with the plane of xy, whose tangent $=\pm\sqrt{\left(\frac{a^2-b^2}{a^2+b^2}\right)}$, is a circle; and the same is therefore true of all sections which are parallel to it.

252. It follows from the last Article, that this hyperboloid may be generated by a circle of variable radius, moving parallel to itself, whose centre moves in a diameter in the plane of the section through $2b$ and $2c$, whose plane makes an

angle $=\tan^{-1}\pm\sqrt{\left(\frac{a^2-b^2}{a^2+b^2}\right)}$, with the axis $2b$, and whose circumference is constantly touched by the principal section in the plane of yz.

253. If we suppose x, y, and therefore z, to be *infinite*, the equation of this *hyperboloid* will coincide with that of the conical surface expressed by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0:$$

its opposite vertices are in the centre, its sections parallel to the plane of xy are ellipses, and it embraces all the linear asymptotes which correspond to the different hyperbolic sections of the hyperboloid.

254. This hyperboloid has *four* summits or vertices at the extremities of the axes $2a$ and $2b$: the axis $2c$ does not terminate in the surface.

255. The hyperboloid of four summits becomes a surface of revolution if $a=b$, the axis of revolution being the axis $2c$; in this case, all the sections parallel to the plane of xy are circles.

256. The *hyperboloid of two sheets or surfaces* consists of two infinite surfaces, whose *two* summits are at the extremities of the axis $2a$: the other two axes, $2b$ and $2c$, do not terminate in the surface of the hyperboloid.

257. There is no section of this hyperboloid in the plane of yz, but the *real* sections which are parallel to it are ellipses: the sections in the plane of xy and xz, as well as all sections parallel to them, are hyperbolas; all parallel sections, whether ellipses or hyperbolas, are similar to each other.

258. The section of this hyperboloid, which passes through the *mean* (Art. 244.) axis $2b$, and which makes with the plane

of xy an angle, whose tangent $= \pm \frac{c}{a}\sqrt{\left(\frac{a^2+b^2}{b^2-c^2}\right)}$, is a circle: and the same is therefore true of all real sections which are parallel to it.

259. It follows from the last Article, that the hyperboloid of two summits may be generated by a circle of variable radius, moving parallel to itself, whose centre moves in a diameter in the plane of the axes $2a$ and $2c$, whose plane makes an angle $= \tan^{-1} \pm \frac{c}{a}\sqrt{\left(\frac{a^2+b^2}{b^2-c^2}\right)}$, with the axis $2a$, and whose circumference is constantly touched by the principal section in the plane of xz.

260. The *hyperboloid of two summits* becomes a surface of revolution if $c = b$, the axis of revolution being $2a$: in this case, all the real sections parallel to the plane of $2b$ and $2c$ are circles.

261. The conical surface expressed by the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0,$$

includes all the linear asymptotes of the corresponding hyperbolic sections of the surface.

SECT. 17.

On Surfaces of the second order which have no centre.

262. It has been shewn in Art. 221, that the terms

$$2a''x, \quad 2b''y, \quad 2c''z,$$

in the general equation of surfaces of the second order, cannot be exterminated *simultaneously*, whenever

$$abc + 2a'b'c' = aa'^2 + bb'^2 + cc'^2 \ldots\ldots\ldots (1).$$

263. It has been shewn, that the terms

$$2a'xy, \quad 2b'xz, \quad 2c'yz,$$

in the same general equation, may be, in all cases exterminated *simultaneously*, without any reference to the values of a, b, c or a'', b'', c'' and d.

264. If the terms involving the products of x, y, z be exterminated simultaneously, in any case where the terms involving the simple powers of x, y, z *cannot* be exterminated simultaneously, then the first transformation must have reduced to zero the coefficient of one *at least* of the terms involving the squares of x, y or z.

For if the transformed equation be

$$ax^2+by^2+cz+dx+ey+fz+k=0 \ldots\ldots\ldots (2),$$

it may, in all cases where a, b, c have finite values, be transformed, into

$$ax^2+by'^2+cz'^2+k=0,$$

by making

$$x=x'-\frac{d}{2a}, \quad y=y'-\frac{e}{2b}, \quad z=z'-\frac{f}{2c},$$

in which case the surface will have a centre: if, however, one or more of the quantities a, b, c be equal to zero, this second transformation is no longer practicable.

265. The general form, therefore, of the equation of surfaces *without centres*, is

$$by^2+cz^2+dx+ey+fz+k=0 \ldots\ldots\ldots (3),$$

where x, y, z are convertible with each other.

266. The equation (3) may be transformed into

$$by'^2+cz'^2+dx'=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

by making

$$y=y'+\beta=y'-\frac{e}{2b},$$

$$z = z' + \gamma = z' - \frac{f}{2c},$$

$$x = x' + \alpha = x' + \frac{ce^2 + bf^2 - 4bck}{4bcd}.$$

267. The curve surface represented by equation (4) is called by the general name of *paraboloid*: it remains to distribute the different surfaces which it includes into their proper classes.

268. Let the coefficients b, c, d, have the same signs. The equation expresses a point which coincides with the origin.

269. Let b and c have the same sign, which is different from that of d.

The principal sections of this surface are parabolas in the planes xy and xz; a point in the plane of yz: but all sections parallel to this plane are *similar* ellipses.

This surface is called the *elliptic paraboloid*.

270. If $b = c$, the sections parallel to the plane of yz become circles, and the paraboloid becomes a surface of revolution round the axis of x.

271. If the parabola whose equation is

$$z^2 = -\frac{d}{c}\,x,$$

move parallel to itself, so that its vertex may trace out the parabola whose equation is

$$y^2 = -\frac{d}{b}\,x,$$

it will generate the elliptic paraboloid

272. If b be greater than c, the section made by a plane passing through the axis of y and making with the plane

of xy an angle $= \sin^{-1} \pm \sqrt{\frac{c}{b}}$, will be a circle: and all sections parallel to it will be circles likewise.

273. If b be less than c, the section made by a plane passing through the axis of z and making with the plane of xz an angle $\pm \sin^{-1} \pm \sqrt{\frac{b}{c}}$, will be a circle: and all sections parallel to it will be circles likewise.

274. In all cases, therefore, an elliptic paraboloid may be generated by a circle of variable radius, moving parallel to itself.

275. Let b and c have different signs.

The principal sections of the corresponding surface are, a parabola $y^2 = -\frac{d}{b}x$, in the plane of xy; a parabola $z^2 = \frac{d}{c}x$, in the plane of xz; two straight lines whose equations are $y = \pm z\sqrt{\frac{c}{b}}$, in the plane of yz: but all sections parallel to the plane of yz and upon both sides of it, are hyperbolas, those at equal distances upon opposite sides of it being conjugate to each other.

This surface is called the *hyperbolic paraboloid*.

276. If the parabola whose equation is

$$z^2 = \frac{d}{c}x,$$

move parallel to itself, so that its vertex may trace out the parabola, whose equation is

$$y^2 = -\frac{d}{b}x,$$

it will generate the hyperbolic paraboloid.

277. The hyperbolic paraboloid can never become a surface of revolution.

278. There is no elliptic and therefore no circular section of the hyperbolic paraboloid, which cannot therefore be generated by the movement of a circle of variable radius parallel to itself.

SECT. 18.

Remarks upon the Interpretation of the Equations of Curves, for those values of the co-ordinates which involve $\sqrt{-1}$.

279. In the preceding Sections of this Chapter, we have limited our interpretations of the equations of curves, to those values of the co-ordinates which were confined to the plane of xy: or in other words, to those values of the co-ordinates which do not involve $\sqrt{-1}$; it remains to consider whether any meaning can be assigned to those values of the co-ordinates which do involve the sign $\sqrt{-1}$.

280. As an example, let us consider the equation of the circle

$$x^2 + y^2 = r^2,$$

where the origin is the centre.

The circle whose radius is r, is the only part of the curve expressed by this equation, which is in the plane of xy.

Let x exceed r: then

$$y = \sqrt{(r^2 - x^2)} = \sqrt{-1}\ \sqrt{(x^2 - r^2)};$$

the values of y affected with the sign $\sqrt{-1}$, *are at right angles* to those values of y which are *not* affected with the sign $\sqrt{-1}$.

If we suppose, therefore, a third axis z, to be drawn at right angles to those of x and y; and if the values of y affected with $\sqrt{-1}$ be reckoned in the plane of xz, at right angles to the corresponding values of y, they will express a curve, which is identical with that which would be expressed by the equation

$$z^2 = x^2 - r^2,$$

in the plane z, x: this curve is an equilateral hyperbola.

In a similar manner, if values of y be taken, in its proper axis, which exceed r, we get

$$x = \sqrt{r^2 - y^2} = \sqrt{-1}\sqrt{(y^2 - r^2)};$$

the corresponding values of x affected with the sign $\sqrt{-1}$ are at right angles to those of x, which are not affected with this sign: the corresponding curve, therefore, will be an equilateral hyperbola in the plane of yz, which is equivalent to that which would be represented by $z^2 = r^2 - y^2$ in its proper plane.

It follows, therefore, that the equation $x^2 + y^2 = r^2$, considered in all its generality, will express a circle in the primitive plane of xy, and two equal equilateral hyperbolas in planes which are at right angles to each other and to the plane of xy.

281. Let it be required to determine the complete system of curves expressed by the equation

$$a^2y^2 + b^2x^2 = a^2b^2.$$

First: An ellipse in the plane of xy; whose semiaxes are a and b.

Secondly: An hyperbola in the plane of xz, equivalent to that which would be expressed in its proper plane by the equation

$$a^2z^2 - b^2x^2 = -a^2b^2.$$

Thirdly: An hyperbola in the plane of yz, which is equivalent to that which would be expressed by the equation

$$a^2y^2 - b^2z^2 = a^2b^2.$$

These three curves have the same axes and origin, and the hyperbolas are conjugate to each other.

282. Let it be required to determine the complete system of curves expressed by the equation

$$y^2 = ax.$$

First: A parabola in the plane of xy, corresponding to positive values of x.

Secondly: A parabola in the plane of xz, corresponding to negative values of x, such as would be expressed, in its proper plane, by the equation

$$-z^2=ax.$$

283. Let it be required to determine the complete system of curves, which are expressed by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=-1.$$

First: There is no curve whatever in the plane of xy.

Secondly: An hyperbola in the plane of zx, equivalent to that which would be expressed, in its proper plane, by the equation

$$a^2z^2-b^2x^2=a^2b^2.$$

Thirdly: An hyperbola in the plane of yz, equivalent to that which would be expressed, in its proper plane, by the equation

$$b^2z^2-a^2y^2=a^2b^2.$$

The values of x and y considered in the preceding Articles, are such as are obtained by the solution of the primitive equation, with respect to x or y, and then supposing them to possess unlimited real values, and *real values only*, whether $+$ or $-$: we are not at liberty to suppose the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=-1$ to become $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, by replacing x by $x\sqrt{-1}$ and y by $y\sqrt{-1}$.

It is obvious that no interpretation can be given upon these principles to those co-ordinates in the equations of *curve surfaces*, which are affected with the sign $\sqrt{-1}$.

CHAP. XII.

ON CERTAIN GONIOMETRICAL SERIES.

1. THERE are many goniometrical series involving powers or multiples of angles, which are extremely useful in many important enquiries: it is proposed in this Chapter to give some of these series, which do not require the aid of the Differential or Integral Calculus or of the Calculus of Finite Differences for their investigation.

2. To find a series for $(2\cos\theta)^m$ in terms of the cosines or sines of multiples of θ.

Let ρ be the arithmetical value of $(2\cos\theta)^m$, (Art. 5. Chap. VIII.) and therefore $(2\cos\theta)^m = 1^m . \rho$, if $2\cos\theta$ be positive, and $= (-1)^m\rho$, if $2\cos\theta$ be negative.

Then, in the first case,

$$(\cos 2mr\pi + \sqrt{-1} . \sin 2mr\pi)\,\rho = \cos m\,(2r\pi + \theta)$$

$$+ m\cos(m-2)\,(2r\pi+\theta) + \frac{m\,(m-1)}{1.2}\cos(m-4)\,(2r\pi+\theta) + \&c.$$

$$+ \sqrt{-1} . \{\sin m\,(2r\pi+\theta) + m\sin(m-2)\,(2r\pi+\theta)$$

$$+ \frac{m\,(m-1)}{1.2}\sin(m-4)\,(2r\pi+\theta)\} = C_r + S_r\sqrt{-1},$$

denoting the series of cosines by C_r and the series of sines by S_r.

In the second case,

$$\{\cos(2r+1)\,m\pi + \sqrt{-1}\sin(2r+1)\,m\pi\}\,\rho = C_r + S_r\sqrt{-1}.$$

3. If m be a whole number, we may make r equal to zero in both members of the equation:

for $C_r = C_0$ and $S_r = S_0$.

If $2\cos\theta$ be positive, $\rho = C_0$.

If $2\cos\theta$ be negative and m even, $\rho = C_0$.

If $2\cos\theta$ be negative and m odd, $\rho = -C_0$.

In all these cases, the value of ρ is expressed therefore by a series of cosines, the corresponding series of sines being equal to zero.

The $(p+1)^{\text{th}}$ term of C_0

$$= \frac{m(m-1)\ldots(m-p+1)}{1.2\ldots.p}\cos(m-2p)\,\theta = T\cos(m-2p)\,\theta.$$

The $(m-p)^{\text{th}}$ term of C_0

$$= T\cos(m-2m+2p)\,\theta = T\cos(m-2p)\,\theta.$$

The terms of the series C_0 being therefore the same from the beginning to the end, it follows that

$$C_0 = 2\left\{\cos m\theta + m\cos(m-2)\,\theta + \frac{m(m-1)}{1.2}\cos(m-4)\,\theta + \&c.\right\}$$

continued to $\frac{m}{2}+1$ terms, when m is even, and to $\frac{m+1}{2}$ terms, when m is odd; the last term in the first case being

$$= \frac{1\times3\times5\ldots(m-1)}{1\times2\times3\ldots\frac{m}{2}}\,2^{\frac{m}{2}-1},$$ [*]

which is the half of the middle term of the binomial $(1+1)^m$; and in the second case being

$$= \frac{1\times3\times5\times\ldots m}{1\times2\times3\times\ldots\frac{m+1}{2}}\times2^{\frac{m-1}{2}}\cos\theta.$$

4. Let m be not a whole number.

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[*] *Algebra*, p. 259.

If $2\cos\theta$ be positive, we find

$$\rho = \frac{C_r}{\cos 2mr\pi} = \frac{S_r}{\sin 2mr\pi}.$$

If $2\cos\theta$ be negative, we find

$$\rho = \frac{C_r}{\cos m(2r+1)\pi} = \frac{S_r}{\sin m(2r+1)\pi}.$$

In both cases, ρ is expressible either by a series of cosines or of sines, unless $\cos 2mr\pi = 0$ or $\sin 2mr\pi = 0$ in one case, and $\cos m(2r+1)\pi = 0$ or $\sin m(2r+1)\pi = 0$ in the other.

In examining such cases, we may suppose m a rational fraction of the form $\frac{p}{n}$ (in its lowest terms), and that r does not exceed $n-1$. (See Art. 2. Chap. VIII.)

5. $\text{Cos}\,\frac{2pr\pi}{n} = 0$, when p is odd, n divisible by 4 or *pariter par*, and $r = \frac{n}{4}$ or $\frac{3n}{4}$.

In this case, $\rho = S_{\frac{n}{2}}$ or $-S_{\frac{3n}{2}}$, if p be of the form $4i+1$ or *pariter impar*, and $\rho = -S_{\frac{n}{2}}$ or $S_{\frac{3n}{2}}$, if p be of the form $4i+3$ or *impariter impar*.

$\text{Sin}\,\frac{2pr\pi}{n} = 0$, when $r = 0$, or when p is odd, n even and $r = \frac{n}{2}$.

In this case, $\rho = C_0$ or $-C_{\frac{n}{2}}$

6. $\text{Cos}\,\frac{p(2r+1)\pi}{n} = 0$, when p is odd, n even and of the form $4i+2$ or *impariter par*, and when $r = \frac{n-2}{4}$ or $\frac{3n-2}{4}$.

In this case, $\rho = S_{\frac{n-2}{4}}$ when p is *pariter impar*, and $-S_{\frac{n-2}{4}}$ when p is *impariter impar*: or $\rho = -S_{\frac{3n-2}{4}}$ or $S_{\frac{3n-2}{4}}$, according as p is *pariter* or *impariter impar*.

$$\text{Sin}\,\frac{p(2r+1)\pi}{n} = 0, \text{ when } n \text{ is odd and } r = \frac{n-1}{2}.$$

In this case, $\rho = C_{\frac{n-1}{2}}$ or $-C_{\frac{n-1}{2}}$, according as p is even or odd.

7. To find a series for $(2 \sin \theta)^m$ in terms of the cosines or sines of the multiples of θ.

$$\begin{aligned}(2\sqrt{-1}\sin\theta)^m &= (e^{\theta\sqrt{-1}} - e^{-\theta\sqrt{-1}})^m \\ &= \cos m(2r\pi+\theta) - m\cos(m-2)(2r\pi+\theta) \\ &\quad + \frac{m(m-1)}{1.2}\cos(m-4)(2r\pi+\theta) + \&\text{c.} \\ &\quad + \sqrt{-1}\,\Big\{\sin m(2r\pi+\theta) - m\sin(m-2)(2r\pi+\theta) \\ &\quad + \frac{m(m-1)}{1.2}\sin(m-4)(2r\pi+\theta) - \&\text{c.}\Big\} = c_r + s_r\sqrt{-1},\end{aligned}$$

if c_r be taken to represent the series of cosines and s_r the corresponding series of sines.

If ρ denote the arithmetical value of $(2\sqrt{-1}\sin\theta)^m$, then, if $2\sin\theta$ be positive, we get

$$\left\{\cos m\left(2r+\frac{1}{2}\right)\pi + \sqrt{-1}\sin m\left(2r+\frac{1}{2}\right)\pi\right\}\rho = c_r + s_r\sqrt{-1};$$

and if $2\sin\theta$ be negative,

$$\left\{\cos m\left(2r+\frac{3}{2}\right)\pi+\sqrt{-1}\sin m\left(2r+\frac{3}{2}\right)\pi\right\}\rho=c_r+s_r\sqrt{-1}.$$

8. If m be a whole number and *even*, then

$$2^{m-1}(\sin\theta)^m=\pm\left\{\cos m\theta-m\,.\cos(m-2)\theta\right.$$
$$\left.+\frac{m(m-1)}{1\,.\,2}\cos(m-4)\theta+\text{\&c. to }\frac{m}{2}\text{ terms}\right\}$$
$$+\frac{1\,.\,3\,.\,5\ldots(m-1)}{1\,.\,2\ldots\ldots\frac{m}{2}}\times 2^{\frac{m}{2}-1},$$

the + or − sign being used according as m is of the form $4i$ or $4i+2$.

If m be a whole number and *odd*, then

$$2^{m-1}(\sin\theta)^m=\pm\left\{\sin m\theta-m\,.\sin(m-2)\theta\right.$$
$$\left.+\frac{m(m-1)}{1\,.\,2}\sin(m-4)\theta-\text{\&c. to }\frac{m+1}{2}\text{ terms}\right),$$

the + or − sign being used according as m is of the form $4i+1$ or $4i+3$.

9. If m be not a whole number, and $2\sin\theta$ positive, then

$$\rho=\frac{c_r}{\cos m\left(2r+\frac{1}{2}\right)\pi}=\frac{s_r}{\sin m\left(2r+\frac{1}{2}\right)\pi}:$$

or its value is expressible either by a series of cosines or of sines, except under the following circumstances.

10. If $m=\frac{p}{n}$ (in its lowest terms), p and n being odd numbers; then

$$\rho=\pm s_{\frac{n-1}{4}},\text{ when } n \text{ is } \textit{pariter impar},$$

the + or − sign being used according as p is *pariter* or *impariter impar*.

If p be even, n *pariter impar*, then

$$\rho = \pm c_{\left(\frac{n-1}{4}\right)},$$

the + or − sign being used according as p is *pariter* or *impariter par*.

11. If m be not a whole number, and $2 \sin\theta$ negative, then

$$\rho = \frac{c_r}{\cos m\,(4r-1)\,\frac{\pi}{2}} = \frac{s_r}{\sin m\,(4r-1)\,\frac{\pi}{2}},$$

its value being expressible either by a series of sines or of cosines, except under the following circumstances.

12. If $m = \frac{p}{n}$ (in its lowest terms), p and n being odd numbers: then

$$\rho = \pm s_{\frac{n+1}{4}} \text{ or } \mp s_{\frac{3n+1}{4}}, \text{ when } n \text{ is } \textit{impariter impar},$$

the + or − sign being used according as p is *pariter* or *impariter impar* in the first case, or the contrary in the second.

If p be even, n *impariter impar*, then

$$\rho = \pm c_{\frac{n+1}{4}} \text{ or } \mp c_{\frac{3n+1}{4}},$$

the + or − sign being used according as p is *pariter* or *impariter impar* in the first case, and the contrary in the second.

13. To express $\cos m\theta$ in terms of a series of ascending powers of $\cos\theta$. Make

$$X = 1 = \frac{m^2}{1.2}\cos^2\theta + \frac{m^2(m^2-2^2)}{1.2.3.4}\cos^4\theta - \frac{m^2(m^2-2^2)(m^2-4^2)}{1.2.3.4.5.6}\cos^6\theta + \&c.$$

$$X' = \cos\theta - \frac{(m^2-1^2)}{1.2.3}\cos^3\theta + \frac{(m^2-1^2)(m^2-3^2)}{1.2.3.4.5}\cos^5\theta - \&c.$$

then

$$\cos m\,(2r\pi+\theta)=\cos m\,(2r+\tfrac{1}{2})\,\pi\times X+\cos\,(m-1)\,(2r+\tfrac{1}{2})\,\pi\times X'.$$

The investigation of the series X and X', cannot *easily* be effected without the aid of the Differential or Integral Calculus, or of the Calculus of Finite Differences: see *Lardner's Trigonometry*, p. 248, for a complete investigation of these and similar series, and of the circumstances under which the coefficients of X or of X' become equal to zero.

14. Given the equation $\tan B = \dfrac{b \sin C}{a - b \cos c}$: to express B in a series proceeding according to sines of the multiples of C.

Replace $\tan B$, $\sin C$ and $\cos C$ by their corresponding exponential expressions (Art. 20, 21, Chap. IX.), and we get

$$e^{2B\sqrt{-}} = \frac{a - b\,.\,e^{-C\sqrt{-1}}}{a - b\,.\,e^{C\sqrt{-1}}};$$

and therefore

$$2B\sqrt{-1} = \log\,(a - b\;\; e^{-C\sqrt{-1}}) - \log\,(a - b\,.\,e^{C\sqrt{-1}})$$

$$= \frac{b}{a}\,(e^{C\sqrt{-1}} - {}^{-C\sqrt{-1}}) + \frac{b^2}{2a^2}\,(e^{2C\sqrt{\;1}} - e^{-2C\sqrt{-1}})$$

$$+ \frac{b^2}{3a^3}\,(e^{3C\sqrt{-1}} - e^{-3C\sqrt{-1}}) + \&c.$$

$$\text{or } B = \frac{b}{a}\sin C + \frac{b^2}{2a^2}\sin 2C + \frac{b^3}{3a^3}\sin 3C + \&c.$$

This is the expression for an angle of a triangle in terms of the angle and side opposite, and the side included between them.

15. Given the equation $\tan\phi = n\tan\theta$, to express ϕ in terms of θ, and conversely.

Replace $\tan\phi$ and $\tan\theta$ by their exponential values, make $k = \dfrac{n-1}{n+1}$, and we get

$$\phi = \theta + k \sin 2\theta + \frac{k^2}{2} \sin 4\theta + \frac{k^3}{3} \sin 6\theta + \&c$$

and, conversely,

$$\theta = \phi - k \sin 2\phi + \frac{k^2}{2} \sin 4\phi - \frac{k^3}{3} \sin 6\phi + \&c.$$

If $n = \cos a$, then $k = \tan^2 \frac{a}{2}$, and therefore

$$\phi = \theta + \tan^2 \frac{a}{2} \sin 2\theta + \tan^4 \frac{a}{2} \sin^4 \theta + \&c.$$

These series are of great importance in many applications of analysis.

16. To expand $(a^2 - 2ab \cos \theta + b^2)^n$, in a series proceeding by cosines of multiples of θ.

If we represent the coefficients of the series for $(1+x)^n$ by C_1, C_2, C_3, ... C_n, &c, then

$$\begin{aligned}
&(a^2 - 2ab \cos \theta + b^2)^n \\
&= a^{2n} \left(1 - \frac{b}{a} e^{\theta\sqrt{-1}}\right)^n \left(1 - \frac{b}{a} e^{-\theta\sqrt{-1}}\right)^n \\
&= \Big\{ a^{2n} \left(1 + C_1^2 . \frac{b^2}{a^2} + C_2^2 . \frac{b^4}{a^4} + C_3^2 . \frac{b^6}{a^6} + \&c.\right) \\
&\quad - 2 \left(C_1 . \frac{b}{a} + C_1 . C_2 . \frac{b^3}{a^3} + C_2 . C_3 . \frac{b^5}{a^5} + \&c.\right) \cos \theta \\
&\quad + 2 \left(C_2 . \frac{b^2}{a^2} + C_1 . C_3 . \frac{b^4}{a^4} + C_2 . C_4 . \frac{b^6}{a^6} + \&c.\right) \cos 2\theta \\
&\quad - 2 \left(C_3 . \frac{b^3}{a^3} + C_1 . C_4 . \frac{b^5}{a^5} + C_2 . C_3 . \frac{b^7}{a^7} + \&c.\right) \cos 3\theta + \&c. \Big\}.
\end{aligned}$$

17. To express $\log (1 - n \cos \theta)$ in a series proceeding according to cosines of multiples of θ.

Make $e = \dfrac{n}{1 + \sqrt{(1 - n^2)}}$, and we shall find

$$\log(1 - n\cos\theta) = 2\log.\frac{n}{\sqrt{(1+n)} - \sqrt{(1-n)}}$$
$$- 2\left(e\cos\theta + \frac{e^2}{2}\cos 2\theta + \frac{e^3}{3}\cos 3\theta + \&c.\right)$$

If $a^2 = b^2 + c^2 - 2bc\cos A$, (Art. 8. Chap. IV.), then

$$\log a = \log b - \left(\frac{c}{b}\cos A + \frac{c^2}{2b^2}\cos 3A + \frac{c^3}{3b^3}\cos 3A + \&c.\right)$$

18. The sum of the series (S),

$$\sin(a + b) + \sin(a + 2b) + \ldots \sin(a + nb)$$
$$= \frac{\sin\left\{a + (n-1)\frac{b}{2}\right\}\sin\frac{nb}{2}}{\sin\frac{b}{2}}.$$

19. If $\frac{nb}{2} = m\pi$, where m is a whole number, or if b be any submultiple of 2π or of $2m\pi$, then $S = 0$.

20. The terms of such a series, however far it may be continued, will be *periodic*, the sum of every successive n terms of it being equal to zero.

21. If $n = 5$, $m = 1$; then $b = \frac{2\pi}{5}$, and

$$\sin\left(a + \frac{2\pi}{5}\right) + \sin\left(a + \frac{4\pi}{5}\right) + \sin\left(a + \frac{6\pi}{5}\right)$$
$$+ \sin\left(a + \frac{8\pi}{5}\right) + \sin(a + 2\pi) = 0;$$

or $\sin a + \sin(72^0 + a) + \sin(36^0 - a) - \sin(36 + a)$
$- \sin(72^0 - a) = 0.$

This is the formula of verification given in Art. 13. Chap. V.

22. The sum of the series

$$\cos(a+b) + \cos(a+2b) + \ldots \cos(a+nb)$$

$$= \frac{\cos\left\{a + (n+1)\dfrac{b}{2}\right\} \sin\dfrac{nb}{2}}{\sin\dfrac{b}{2}}.$$

23. The sum of the series

$$\sin(a+b) - \sin(a+2b + \sin(a+3b) - \ldots \pm \sin(a+nb),$$

is $\dfrac{-\cos\left\{a + (n+1)\dfrac{b}{2}\right\} \sin\dfrac{nb}{2}}{\cos\dfrac{b}{2}}$, if n be even,

and $\dfrac{\sin\left\{a + (n+1)\dfrac{b}{2}\right\} \cos\dfrac{nb}{2}}{\cos\dfrac{b}{2}}$, if n be odd.

24. The sum of the series

$$\cos(a+b) - \cos(a+2b) + \ldots \pm \cos(a+nb),$$

is $\dfrac{\sin\left\{a + (n+1)\dfrac{b}{2}\right\} \sin\dfrac{nb}{2}}{\cos\dfrac{b}{2}}$, if n be even,

and $\dfrac{\cos\left\{a + (n+1)\dfrac{b}{2}\right\} \cos\dfrac{nb}{2}}{\cos\dfrac{b}{2}}$, if n be odd.

25. The sum of any of the preceding series continued to *infinitum*, would not be expressed by making n *infinite*, in any one of the expressions for their *sums*.

26. $$x^{2m} - 2\cos\theta\,.\,x^m + 1$$

$$=\left(x^2-2x\cos\frac{\theta}{m}+1\right)\left(x^2-2x\cos\frac{2\pi+\theta}{m}+1\right)\left(x^2-2x\cos\frac{4\pi+\theta}{m}+1\right)$$

$$\ldots\left\{x^2-2x\cos\frac{2(m-1)\pi+\theta}{m}+1\right\}.$$

For $(\cos\theta\pm\sqrt{-1}\sin\theta)^{\frac{1}{m}}=\cos\frac{2r\pi+\theta}{m}\pm\sqrt{-1}\sin\frac{2r\pi+\theta}{m}$,

(where r is any number in the series $0, 1, 2, \ldots m-1$), expresses the different values of x in the equation

$$x^{2m}-2x^m\cos\theta+1=0.$$

27. $$\text{Sin}\,\frac{\theta}{2}=2^{m-1}\sin\frac{\theta}{2m}\,.\,\sin\left(\frac{\theta+2\pi}{2m}\right)\,.\,\sin\left(\frac{\theta+4\pi}{2m}\right)$$

$$\ldots\sin\left\{\frac{\theta+2(m-1)\pi}{2m}\right\}.$$

This follows from the last Article, by making $x=1$, and replacing

$$2-2\cos\theta,\quad 2-2\cos\frac{\theta}{m},\quad 2-2\cos\frac{2\pi+\theta}{m},\ \text{\&c.}$$

$$\text{by}\quad 4\sin^2\frac{\theta}{2},\quad 4\sin^2\frac{\theta}{2m},\quad 4\sin^2\frac{2\pi+\theta}{2m},\ \text{\&c.}$$

28. $$\text{Sin}\,\theta=\theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{4\pi^2}\right)\left(1-\frac{\theta^2}{9\pi^2}\right)\ldots\ in\ inf.$$

29. $$\text{Cos}\,\theta=\left(1-\frac{4\theta^2}{\pi^2}\right)\left(1-\frac{4\theta^2}{9\pi^2}\right)\left(1-\frac{4\theta^2}{25\pi^2}\right)\ldots\ in\ inf.$$

30. $$\theta=2\left\{\sin\theta-\tfrac{1}{2}\sin 2\theta+\tfrac{1}{3}\sin 3\theta-\tfrac{1}{4}\sin 4\theta+\text{\&c.}\right\}$$

31. $$\theta=\sin\theta\sec\frac{\theta}{2}\,.\,\sec\frac{\theta}{2^2}\,.\,\sec\frac{\theta}{2^3}\ldots\ldots\ in\ inf.$$